Introduction to microlocal analysis
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Microlocal analysis was developed as part of the study of partial differential equations by extending notions of analysis from the study of smooth functions to the study of distributions or 'generalized functions'. The presence of singularities brings very delicate questions which have received refined answers. Thus for example the wavefront set of a distribution determines the regularity of a distribution at a given point *in a given direction*. This information allows us to answer some basic questions in distribution theory like 'when can two distributions be multiplied?' but also allows us to strengthen and refine results in partial differential equations. The main tool to apply these results is the theory of pseudodifferential operators, a simultaneous generalization of differentiation and integration. Analyzing the behavior of a pseudodifferential operator on the wave front set of a distribution, for example, we recover and refine propagation of singularities and elliptic regularity.

In this course we will develop the theory of pseudodifferential operators on Euclidean and curved geometries. We will use these operators to construct refined parametrices and inverses of natural differential operators. For example given a Riemannian manifold, the resolvent of the Laplacian, the solution to the wave equation and the solution of the heat equation are all best understood using the techniques of microlocal analysis.

The topics treated in this course will be useful for students looking to do research in mathematical physics, analysis and/or geometry. The prerequisites are a course in geometry including vector bundles and differential forms on manifolds, and a course in analysis including Lp spaces. (A course in topology would be useful but is not necessary.)

A possible syllabus, subject to time constraints:
Pseudodifferential operators on $\mathbb{R}^n$
Wave front set and pseudolocality
Pseudodifferential operators on manifolds
The resolvent of the Laplacian
Elliptic regularity
De Rham cohomology and Hodge theory
Further topics depending on student interest:
Wave equation, Dirac operators, heat equation, Atiyah-Singer index theory, etc.