QUOTIENTS OF COMPACT AND REDUCTIVE GROUPS

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1. Brief outline of topics

Let K be a compact Lie group acting smoothly on a manifold M. Basic objects of study are the quotient M/K and properties of the quotient morphism $\pi\colon M \to M/K$. We will prove the differentiable slice theorem and a theorem about smooth invariants to deduce the structure of M/K and π. Usually M will equivariantly embed into a representation space W of K (for example, if M is compact) and so one can reduce to studying W in place of M. We will establish the result of Hilbert that the algebra $\mathbb{R}[W]^K$ is finitely generated, say by $p_1, \dots, p_d$. Let $p = (p_1, \dots, p_d)\colon W \to \mathbb{R}^d$ and let X denote the image of p (which is semialgebraic). We will show that p induces a homeomorphism $\bar{p}\colon W/K \to X$ and in fact that $\bar{p}$ is a diffeomorphism if one defines $C^\infty(W/K)$ to be $C^\infty(W)^K$ and $C^\infty(X)$ to be the restriction of $C^\infty(\mathbb{R}^d)$ to X [Sch75]. A basic and interesting problem is

(1) Determine the inequalities defining the semialgebraic set X [PS85].

To do this one we will need to look at the action of the complexification $G := K_{\mathbb{C}}$ of K which acts on the complexification $V := W \otimes_{\mathbb{R}} \mathbb{C}$ of W. Thus we are naturally led to consider actions of complex reductive groups. We have a quotient $V/\!\!/G$ (the space of closed orbits) and the mapping p above, considered as a map of V to $\mathbb{C}^d$, has image $Z \simeq V/\!\!/G$. There is a real algebraic subset $\mathcal{M}$ of V, the Kempf-Ness set, such that $p\colon \mathcal{M} \to Z$ is surjective with fibers being K-orbits. We will study the important interplay between the geometry of the G-action and that of K acting on $\mathcal{M}$. This interplay is what will allow us to find the inequalities defining X. Many important theorems concerning the G-action have very transparent proofs using $\mathcal{M}$. For example, this holds for the results of Luna's difficult paper [Lun75].

We will also study arbitrary representations V of complex reductive groups G. An important tool is Luna's slice theorem [Lun73] which allows one to deduce the local structure of $Z := V/\!\!/G$ and the canonical map $\pi\colon V \to Z$. There is a natural finite stratification $Z = \cup Z_{(H)}$ where a closed orbit Gv lies in $Z_{(H)}$ if the conjugacy class of its isotropy groups is the conjugacy class of the subgroup H of G (H is necessarily reductive). There is a unique open stratum, the principal stratum, and the elements of the corresponding conjugacy class (H) are called principal isotropy groups. Then V^H has a dense image in Z. Above each stratum $Z_{(H)}$, the morphism π is a G-fibration.

Some questions we consider, which can be at least partially answered by the techniques we will develop, are the following:

(2) Suppose that the principal isotropy class (H) is not trivial. What can we then say about the invariants and quotient (Luna-Richardson Theorem)?
(3) Let φ be an algebraic (or holomorphic) automorphism of Z. Does φ preserve the stratification of Z? See [KR08, Sch13] ?
(4) Does φ necessarily lift to an automorphism Φ of V which preserves the fibers of π? Can we find a Φ with some kind of equivariance property [Kut11, Sch12]?
(5) Same questions for W/K and smooth automorphisms preserving the stratification.

In sum, we will study important techniques in the theories of compact smooth transformation groups and reductive complex algebraic group actions and will exploit the connections between the two theories to establish fundamental results in both areas as well as to answer interesting questions such as (1)–(5) above.

2. Contents of the Lectures

(1) Lie groups and complex algebraic groups.
(2) Vector fields and Lie algebras. Exponential map.
(3) Group actions and homogeneous spaces.
(4) Representations.
(5) Homomorphisms of Lie algebras and Lie groups.
(6) Principal bundles. Homogeneous vector bundles.
(7) Haar measure and integration over compact groups. Complete reducibility of representations.
(8) Hilbert's theorem for representations of compact groups. Quotients of compact group actions. The smooth invariants theorem.
(9) The differentiable slice theorem. Applications.
(10) Isotropy strata and their properties.
(11) Principal isotropy groups. Luna-Richardson theorem (compact group case).
(12) Maximal tori of compact Lie groups (using the slice theorem).
(13) Peter Weyl theorem. Structure of compact Lie groups. Complexification.
(14) Structure of $K_{\mathbb{C}}$, Cartan decomposition.
(15) Quotients by complex reductive groups (Weyl's unitary trick).
(16) Kempf-Ness theory.
(17) Inequalities defining orbit spaces of compact groups.
(18) Neeman's retraction theorem.
(19) Luna's slice theorem I. A proof.
(20) Luna's slice theorem II. Applications.
(21) Stratification of the quotient space.
(22) Luna-Richardson theorem (complex case) and invariants of the classical representations of the classical groups.
(23) Derivations and differential operators on singular spaces.
(24) The lifting problem (can one lift differential operators from $V/\!\!/G$ to V?) [Sch95].

References

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