Combinatoria di grafi colorati e di stringhe colorate

Enrico Rogora

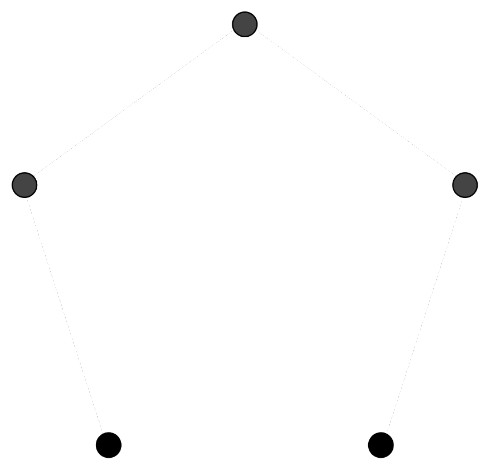
Il principio dei cassetti (di Dirichlet)

Fondamentale si rivela il seguente semplice
PRINCIPIO (detto “di Dirchlet”)
Se n+1 oggetti sono messi in n cassetti, almeno un cassetto deve contenere almeno due oggetti.
CONSEGUENZE
A Roma ci sono almeno due persone (non calve) con lo stesso numero di capelli.

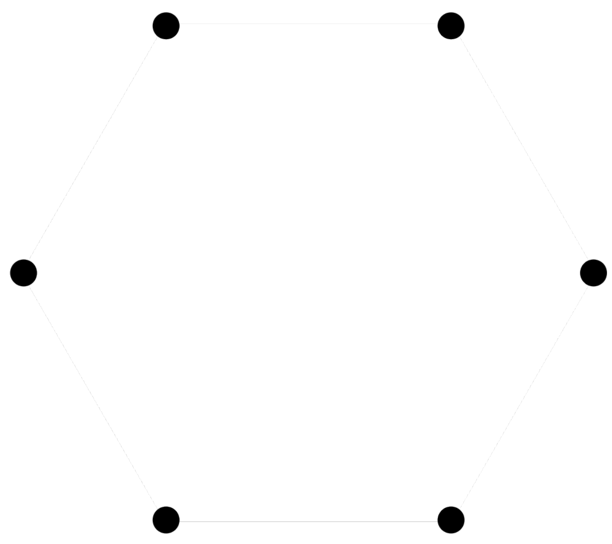
Il gioco di Ramsey

I giocatore “A” comincia congiungendo due vertici del diagramma iniziale con un tratto rosso. Il giocatore “B” risponde congiungendo due vertici del diagramma (non già congunti precedentemente) con un tratto blu. Si ripetono le mosse, sempre mantenendo la condizione che due vertici possono essere congiunti al più da un tratto, fino a che uno dei due giocatori riesce a disegnare un triangolo monocromatico ovvero un triangolo in cui tutti gli spigoli sono dello stesso colore, oppure fino a che tutte le coppie di vertici sono congiunte da uno (e un solo) tratto. Gli spigoli possono intersecarsi ma le intersezioni non producono nuovi vertici.

Ramsey 5

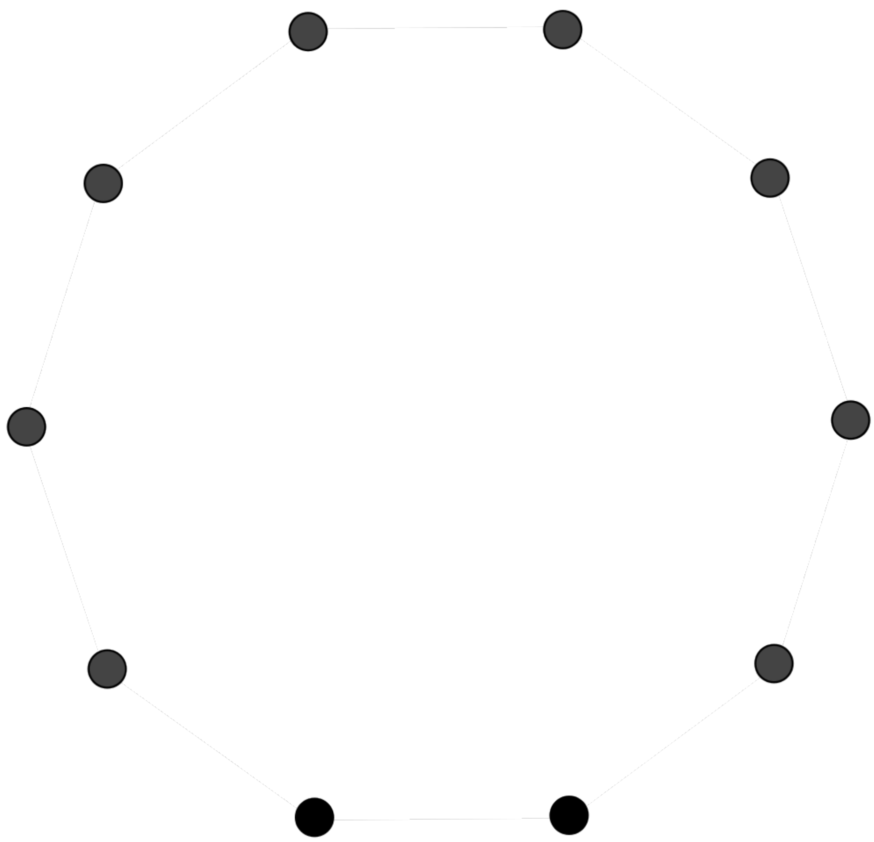


Ramsey 6



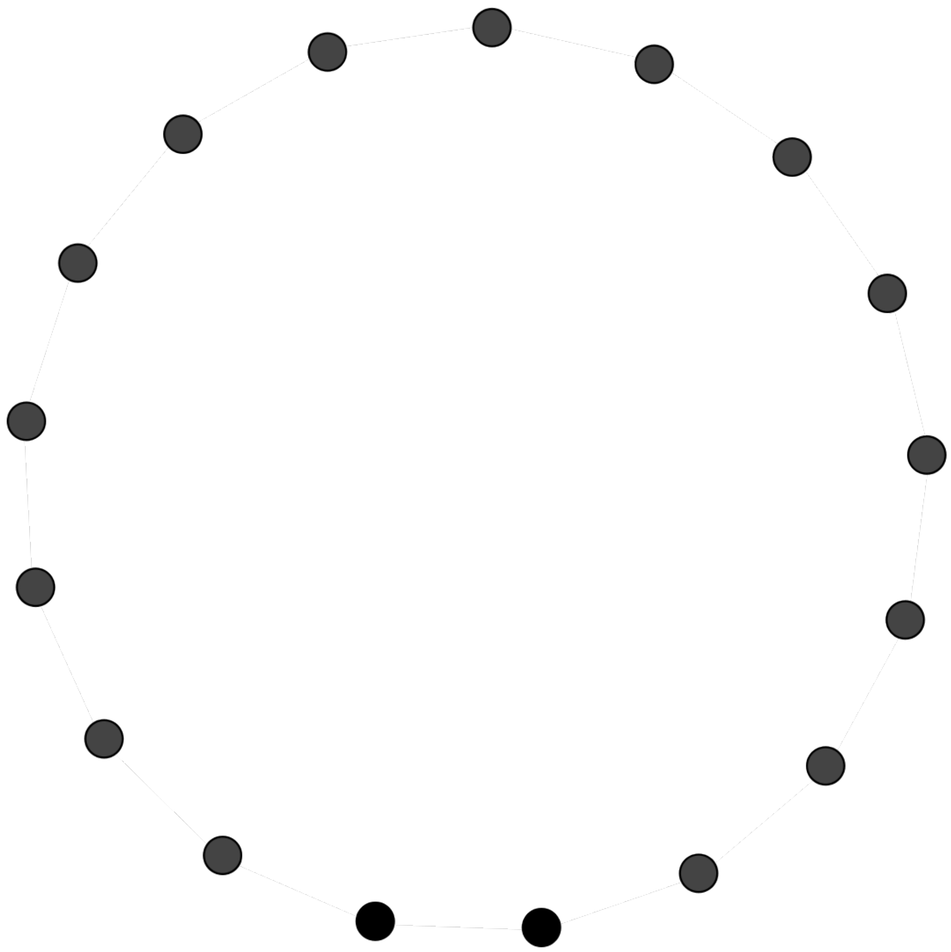
Ramsey 10

Con i triangoli è troppo facile? quale figura moncromatica vogliamo cercare?
Il quadrangolo (definizione: quattro vertici collegati da quattro spigoli tale che ogni vertice sta su due spigoli. Si ammette che gli spigoli possano intrecciarsi):
La 4-clique (4-clicca oppure 4-cricca), cioè il grafo completo su quattro vertici (sei spigoli, ogni vertice sta su tre spigoli). Sperimentare come funziona l'idea di sfidarsi a disegnare un quadrangolo monocromatico su 8 vertici e poi collaborare per colorare un grafo con due colori privo di 4-clique su 10 vertici.



Ramsey 17

E qui? Questo è ingiocabile! Serve però per porre il problema del tempo necessario per colorarlo. Domanda da porre agli studenti: qual è il numero massimo degli spigoli che si possono disegnare? Risposta: 16+15+14+...+2+1=(16)(17)/2=153. Assumendo di impiegare una media di 10 secondi per disegnare un vertice e controllare se è appare una 4-clique, sarebbero necessari 1530 secondi, cioè circa mezz’ora.



Setting astratto grafi e colorazioni.

Grafici, vertici, spigoli, colorazioni, clique, ordine di una clique.
Applicazioni: il problema delle mutue conoscenze.

Teorema di Ramsey (per i grafi colorati con due colori).

Per ogni $k \geq 2$, esiste un qualche intero N (N=N(k)) tale che, ogni due colorazione di un grafo con almeno N vertici contiene un sottografo monocromatico completo (clique) su k vertici.

Chi era Frank Plumpton Ramsey?

Frank Ramsey (1903-1930)'s parents were Arthur Stanley Ramsey and Agnes Mary Wilson. Arthur Ramsey was President of Magdalene College, Cambridge, and a tutor in mathematics there. Frank was the oldest of his parents four children. He had one brother and two sisters and his brother Michael Ramsey went on to become Archbishop of Canterbury.

Ramsey entered Winchester College in 1915 and from there he won a scholarship to Trinity College, Cambridge. He completed his secondary school education at Winchester in 1920 and he entered Trinity College, Cambridge, to study mathematics. At Cambridge, Ramsey became a senior scholar in 1921 and graduated as a Wrangler in the Mathematical Tripos of 1923.

After graduating, Ramsey went to Vienna for a short while, returning to Cambridge where he was elected a fellow of King's College Cambridge in 1924. It is worth noting that this was a most unusual occurrence, and in fact Ramsey was only the second person ever to be elected to a fellowship at King's College, not having previously studied at King's.

In 1925 Ramsey married Lettice C Baker and they had two daughters. In 1926 he was appointed as a university lecturer in mathematics and he later became a Director of Studies in Mathematics at King's College. It was a short career, for sadly Ramsey died at the beginning of 1930. However, in the short time during which he lectured at Cambridge he had already established himself as an outstanding lecturer. Broadbent writes in [1]:-

> *His lectures on the foundations of mathematics impressed young students by their remarkable clarity and enthusiasm ...*

Although Ramsey was a lecturer in mathematics, he produced work in a remarkable range of topics over a short period. As well as starting up the new area of mathematics now called 'Ramsey theory', which we say more about below, he wrote on the foundations of mathematics, economics and philosophy.

He published his first major work *The Foundations of Mathematics* in 1925. In this work he accepted the claim by Russell and Whitehead made in the *Principia Mathematica* that mathematics is a part of logic. Ramsey's aim in this paper, however, was to improve on the *Principia Mathematica* and he did so in two ways. Firstly he proposed dropping the axiom of reducibility which, he writes, is:-

> *... certainly not self-evident and there is no reason to suppose it true; and if it were true, this would be a happy accident and not a logical necessity, for it is not a tautology.*

His second simplification is to suggest simplifying Russell's theory of types by regarding certain semantic paradoxes as linguistic. He accepted Russell's solution to remove the logical paradoxes of set theory arising from, for example, "the set of all sets which are not members of themselves". However, the semantic paradoxes such as "this is a lie" are, Ramsey claims, quite different and depend on the meaning of the word "lie". These he removed with his reinterpretation that removed the axiom of reducibility.

Ramsey published *Mathematical Logic* in the *Mathematical Gazette* in 1926. In this he attacks the:-

> *... Bolshevik menace of Brouwer and Weyl ...*

for denying that propositions are either true of false. He writes:-

> *Brouwer would refuse to agree that either it was raining or it was not raining, unless he had looked to see.*

He also criticises Hilbert in *Mathematical Logic* saying that he had attempted to reduce mathematics to:-

> *... a meaningless game with marks on paper.*

His second paper on mathematics *On a problem of formal logic* was read to the London Mathematical Society on 13 December 1928 and published in the *Proceedings of the London Mathematical Society* in 1930. This examines methods for determining the consistency of a logical formula and it includes some theorems on combinatorics which have led to the study of a whole new area of mathematics called Ramsey theory. Harary describes this birth of Ramsey theory in [8] where he writes the following:-

The celebrated paper of Ramsey [*in* 1930] *has stimulated an enormous study in both graph theory ..., and in other branches of mathematics .... Most certainly 'Ramsey theory' is now an established and growing branch of combinatorics. Its results are often easy to state* (*after they have been found*) *and difficult to prove; they are beautiful when exact, and colourful. Unsolved problems abound, and additional interesting open questions arise faster than solutions to the existing problems.*

The combinatorics was introduced by Ramsey to solve a special case of the decision problem for the first-order predicate calculus. However, as Mellor points out in [9], it is now known that there is a more direct proof than that given by Ramsey, while the general case of the decision problem cannot be solved. So Mellor points out that [9]:-

Ramsey's enduring fame in mathematics ... rests on a theorem he didn't need, proved in the course of trying to do something we now know can't be done!

Ramsey made a systematic attempt to base the mathematical theory of probability on the notion of partial belief. This work on probability, and also important work on economics, came about mainly because Ramsey was a close friend of Keynes. Being a friend of Keynes certainly did not stop Ramsey attacking Keynes' work, however, and in *Truth and probability* , which Ramsey published in 1926, he argues against Keynes' ideas of an a priori inductive logic. Ramsey's arguments convinced Keynes who then abandoned his own ideas. Ramsey, proposing a probability measure based on strength of belief, [11]:-

... derives measures both of desires (*subjective utilities*) *and of beliefs* (*subjective probabilities*)*, thereby founding the now standard use of these concepts.*

In economics, Ramsey wrote two papers *A contribution to the theory of taxation* and *A mathematical theory of saving*. These would lead to important new areas in the subject.

It was philosophy, however, that was Ramsey's real love. He wrote a number of works such as *Universals* (1925), *Facts and propositions* (1927), *Universals of law an of fact*(1928), *Knowledge* (1929), *Theories* (1929), and *General propositions and causality* (1929). Braithwaite writes in [6]:-

... in general philosophy took more and more of his attention. For profitable thought in this most difficult field Ramsey was superbly equipped, and there is no doubt that his early death has deprived the world of one of its most promising philosophers.

One would have to say, however, that Ramsey's work in philosophy has been somewhat overshadowed by that of Wittgenstein. Recently, however, Ramsey's work in philosophy seems to be receiving more attention.

Several of the articles cited in the references paint vivid pictures of Ramsey's character. For example Braithwaite writes in [6]:-

As a person, no less than as a thinker, Ramsey was an ornament to Cambridge. From his undergraduate days he had been recognised as an authority on any abstract subject, and his directness of approach and candour were an inspiration to his associates. His enormous physical size fitted well the range of his intellect, and his devastating laugh suited his power of humorously *discarding irrelevancies, which power enabled him to be both subtle and profound in the highest degree.*

Mellor, in [10], paints a similar picture:-

He was a quiet, modest man, easy going and uninhibited, with a loud infectious laugh, his tolerance and good humour enabling him to disagree strongly without giving or taking offence; as with his brother Michael ... whose ordination ... Frank, as a militant atheist, regretted.

He was tall (six feet three inches) and bulky, short-sighted, wore steel-rimmed spectacles and appeared clumsy but was in fact a good tennis player. He produced his remarkable output in four hours a day - he found it too exacting to do more - in the mornings, with afternoons and evenings often spent walking or listening to records. He listened a lot to classical music, both live and recorded, and was a keen hill-walker.

In [2] his potential is emphasised:-

There was no one in Cambridge among the younger men who would be considered his equal for power and quality of mind, and also for the boldness and originality of conception in one of the most difficult subjects of study.

Ramsey suffered an attack of jaundice and was taken to Guy's Hospital in London for an operation. He died following the operation.

Dimostrazione del teorema di Ramsey

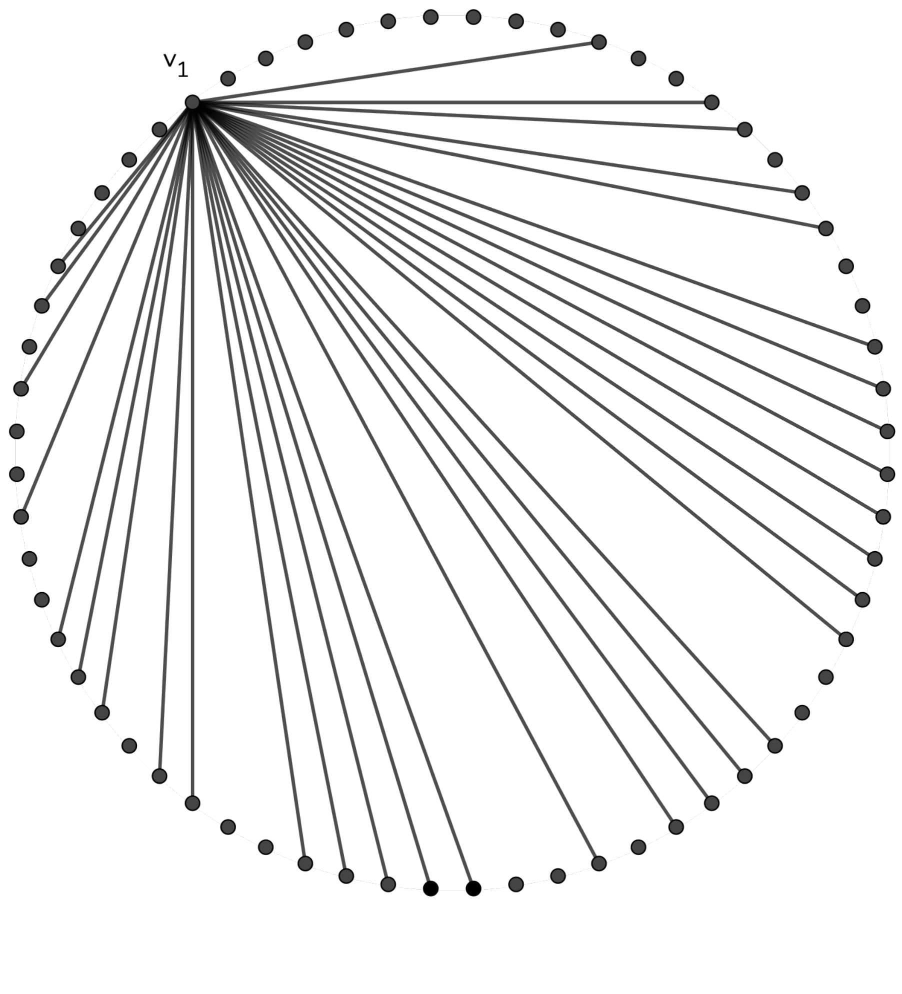
Alcune considerazioni preliminari:
Per n=5, esiste una bicolorazione priva di triangoli monocromatici.
In ogni 6 colorazione esiste un triangolo monocromatico.

Teorema

Esiste sempre almeno una 4 clique monocromatica su ogni grafo con 64 vertici colorato con due colori.

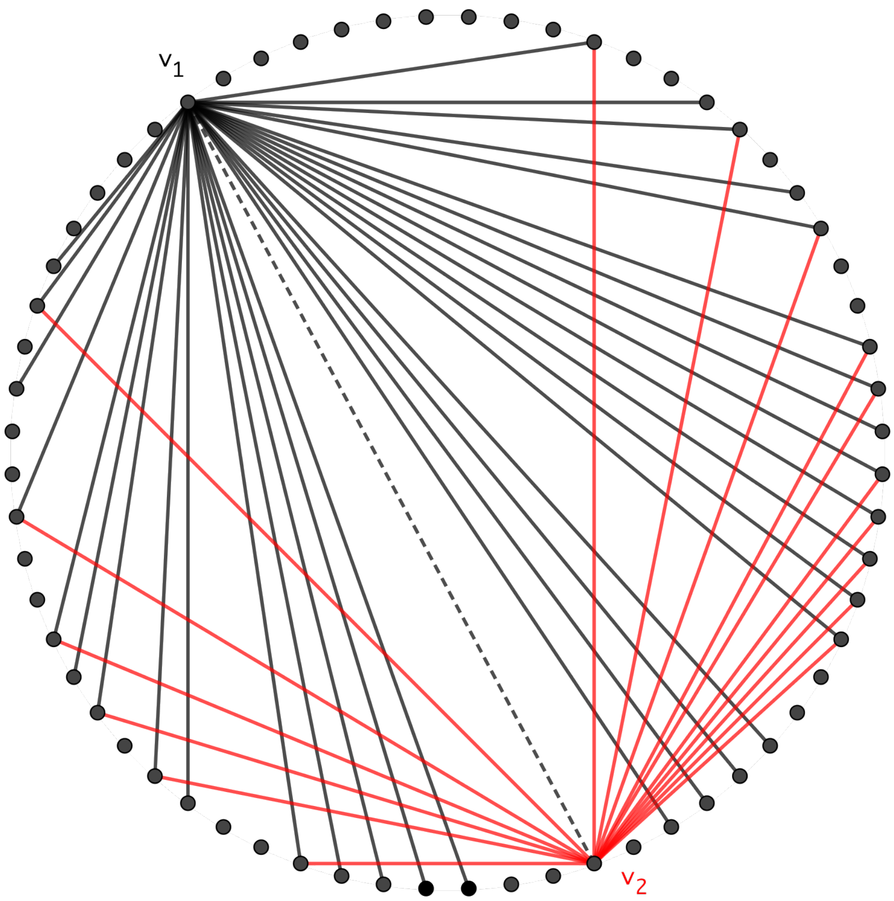
Step 1

Scegli un vertice qualsiasi e chiamalo v1. Da v1 ci sono 63 spigoli che lo collegano agli altri vertici. Scegli 32 spigoli dello stesso colore (esistono sempre per il principio di Dirichlet) e chiamali S1-spigoli. Nell'esempio il "colore prevalente" degli S1-spigoli è il nero. Assegna a v1 il colore degli S1-spigoli, che nell'esempio è il nero (se fosse stato il rosso, l'argomento funzionerebbe lo stesso).



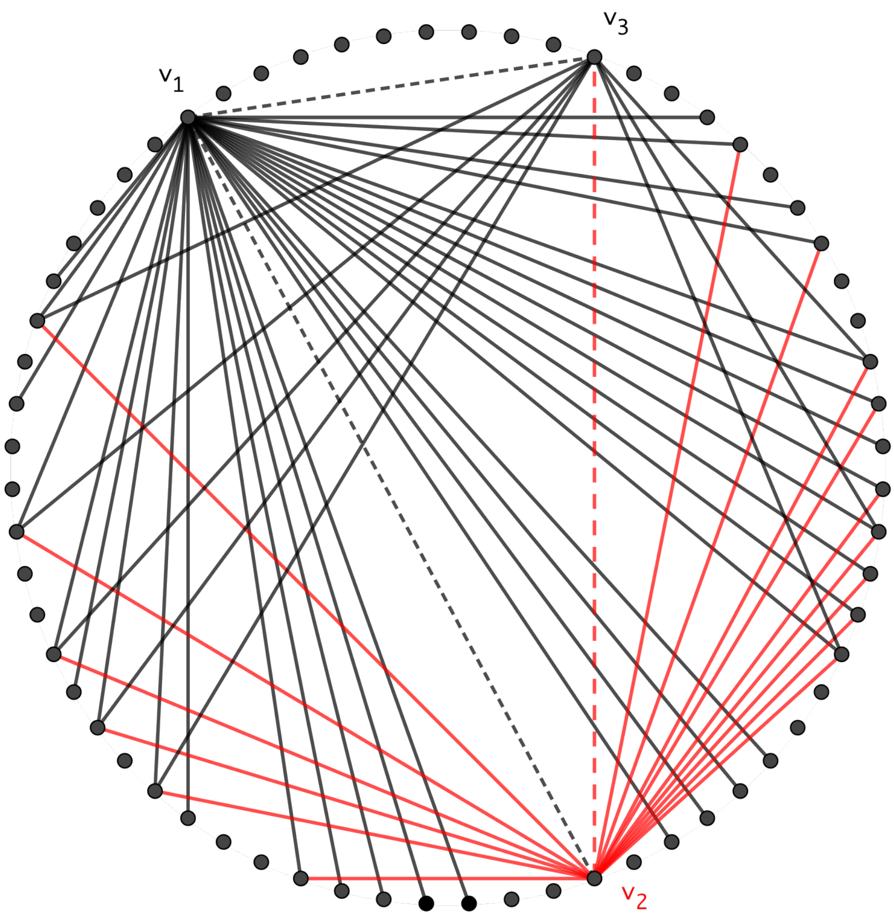
Step 2

Scegli un vertice qualsiasi tra i 32 collegati a v1 con gli S1-spigoli, e chiamalo v2. Da v2 ci sono 31 spigoli che collegano v2 ai vertici degli S1-spigoli diversi da v1. Scegli 16 tra questi 31 spigoli, che siano dello stesso colore (esistono sempre per il principio di Dirichlet) e chiamali S2-spigoli. Nell'esempio il "colore prevalente" degli S2-spigoli è il rosso, ma l'argomento funzionerebbe lo stesso anche se il colore degli S2-spigoli fosse il nero. Assegna a v2 il colore prevalente, che nell'esempio è il rosso.



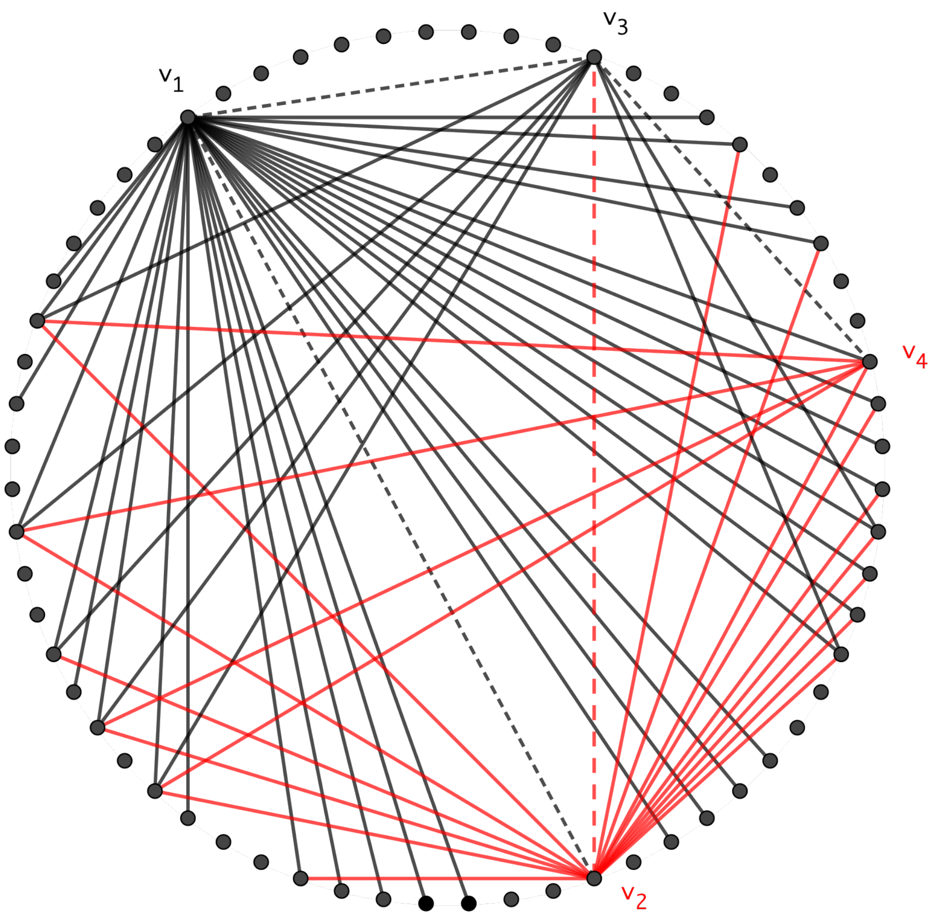
Step 3

Scegli un vertice qualsiasi tra i 16 collegati a v2 con gli S2-spigoli, e chamalo v3. Da v3 ci sono 15 spigoli che lo collegano ai vertici degli S2-spigoli diversi da v2. Scegli 8 tra questi 15 spigoli, che siano dello stesso colore (esistono sempre per il principio di Dirichlet) e chiamali S3-spigoli. Nell'esempio il “colore prevalente” degli S3-spigoli è il nero, ma l'argomento funzionerebbe lo stesso anche se il colore degli S3-spigoli fosse il rosso. Assegna a v3 il colore prevalente, che nell'esempio è il nero.



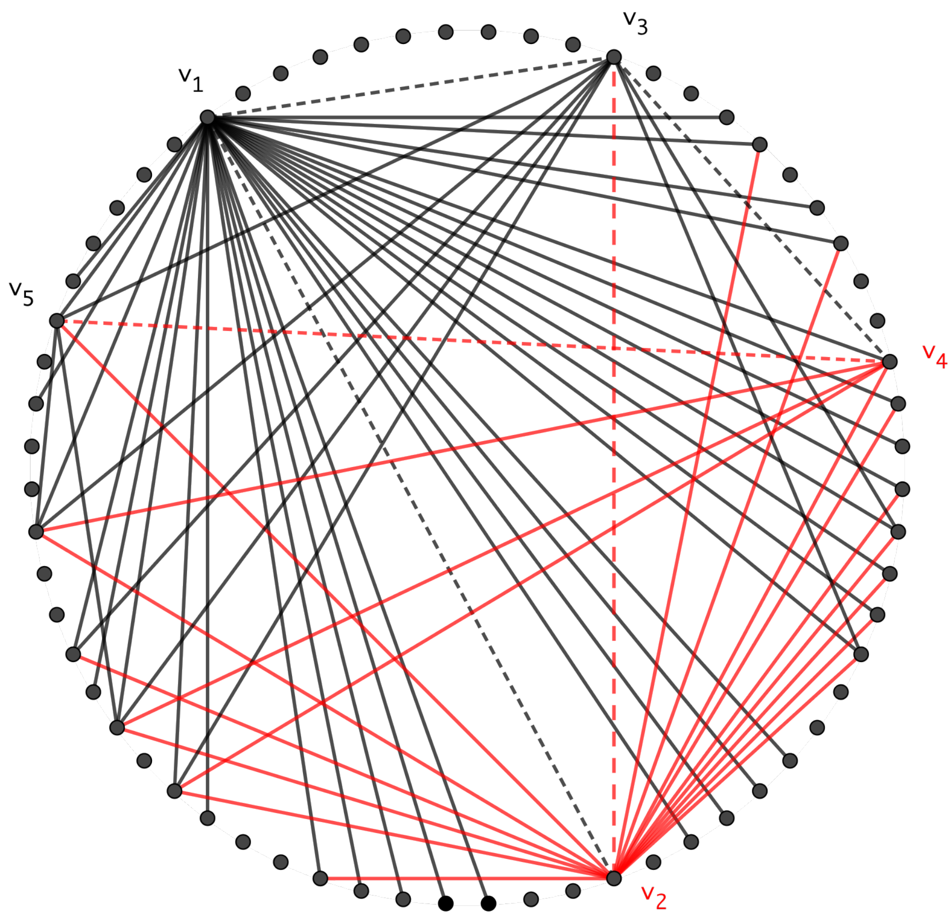
Step 4

Scegli un vertice qualsiasi tra gli 8 collegati a v3 con gli S3-spigoli, e chamalo v4. Da v4 ci sono 7 spigoli che lo collegano ai vertici degli S3-spigoli diversi da v3. Scegli 4 tra questi 7 spigoli, che siano dello stesso colore (esistono sempre per il principio di Dirichlet) e chiamali S4-spigoli. Nell'esempio il "colore prevalente" degli S4-spigoli è il rosso, ma l'argomento funzionerebbe lo stesso anche se il colore degli S4-spigoli fosse il nero. Assegna a v4 il colore prevalente, che nell'esempio è il rosso.



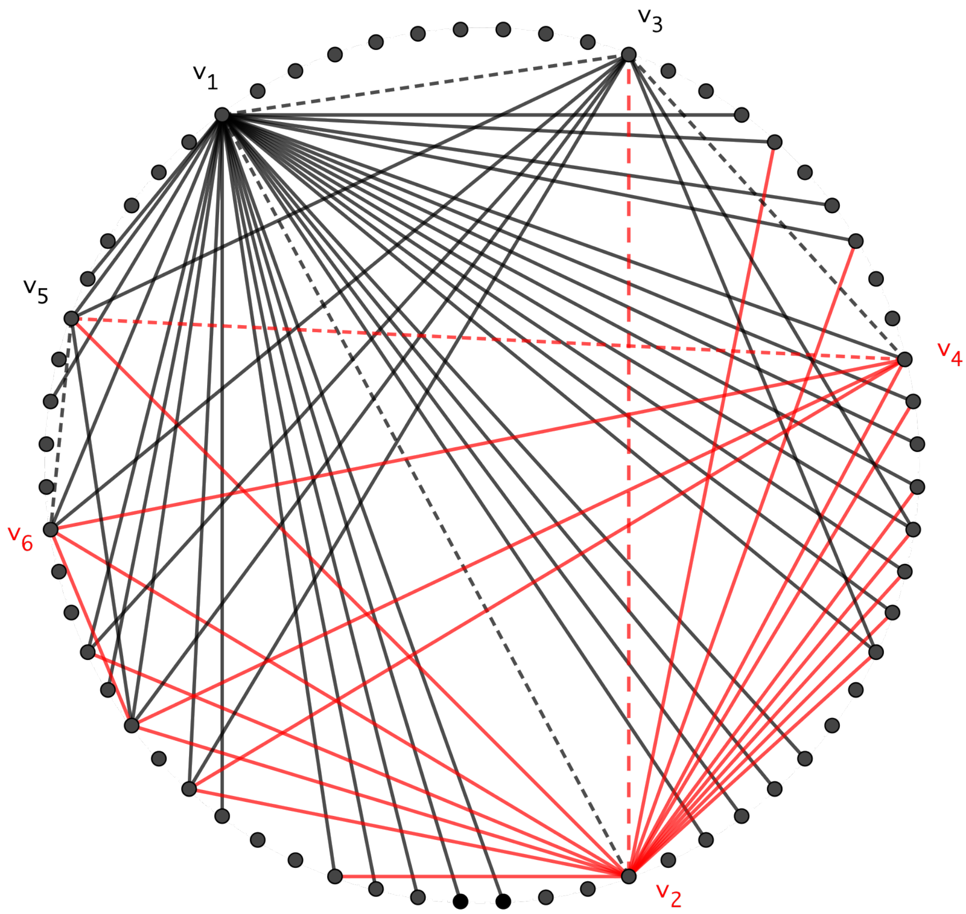
Step 5

Scegli un vertice qualsiasi tra i 4 collegati a v4 con gli S4-spigoli, e chamalo v5. Da v5 ci sono 3 spigoli che lo collegano ai vertici degli S4-spigoli diversi da v4. Scegli 2 tra questi 3 spigoli, che siano dello stesso colore (esistono sempre per il principio di Dirichlet) e chiamali S5-spigoli. Nell'esempio il "colore prevalente" degli S5-spigoli è il nero, ma l'argomento funzionerebbe lo stesso anche se il colore degli S5-spigoli fosse il rosso. Assegna a v5 il colore prevalente, che nell'esempio è il nero.



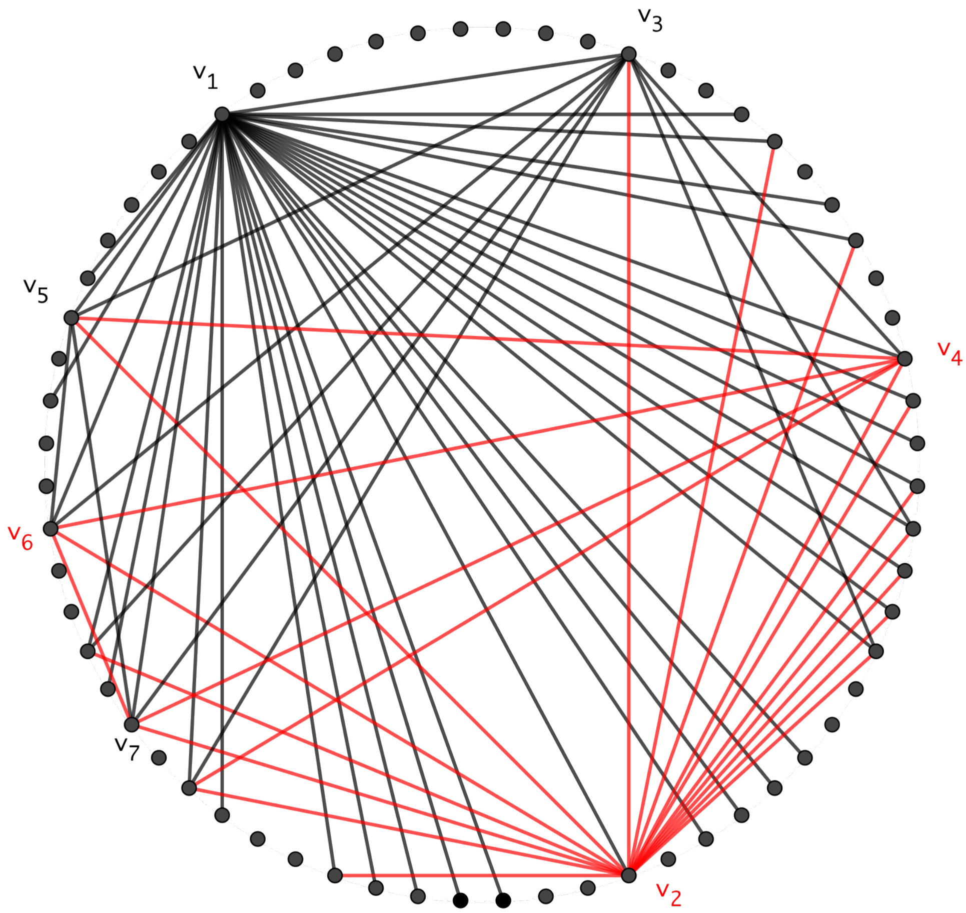
Step 6

Scegli un vertice qualsiasi tra i 2 collegati a v5 con gli S5-spigoli, e chamalo v6. Da v6 c'è un solo spigolo che lo collega ai vertici degli S5-spigoli diversi da v5. Scegli 4 tra questi 7 spigoli, che siano dello stesso colore (esistono sempre per il principio di Dirichlet) e chiamalo S6-spigolo. Nell'esempio il colore dell'S6-spigolo è il rosso, ma l'argomento funzionerebbe anche se il colore fosse il nero. Assegna a v6 il colore dell'S6 spigolo, che nell'esempio è il rosso.



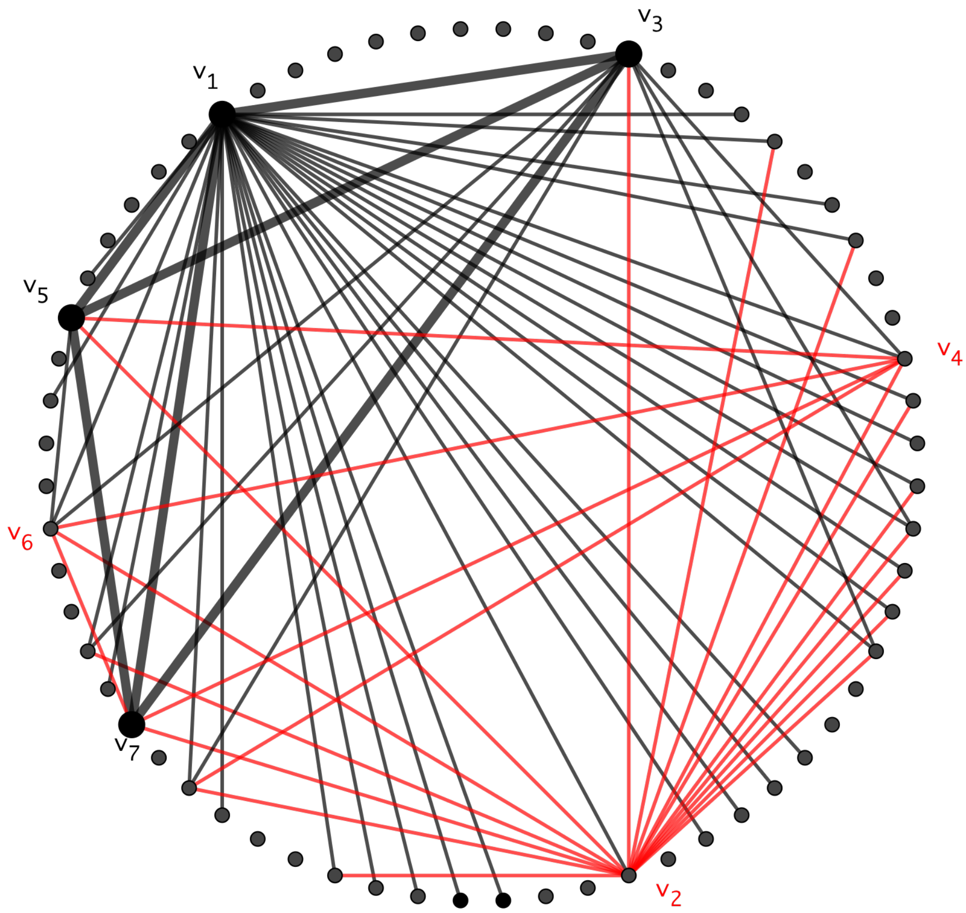
Step 7

Chiamo v7 il vertice dell'S6- spigolo diverso da v6.
I 6 vertici v1,v2,...,v6 sono colorati con due colori. Per il principio di Dirichlet ne ho almeno tre dello stesso colore. Colorando v7 del colore che mi serve, posso garantire di averne quindi sempre almeno quattro dello stesso colore. Nell'esempio v1, v3, e v5 sono neri mentre v2, v4 e v6.
Se scegli di colorare v7 di nero, hai quattro vertici neri: (v1,v3,v5,v7).
Se scegli di colorare v7 di rosso, hai quattro vertici rossi (v2,v4,v6,v7)



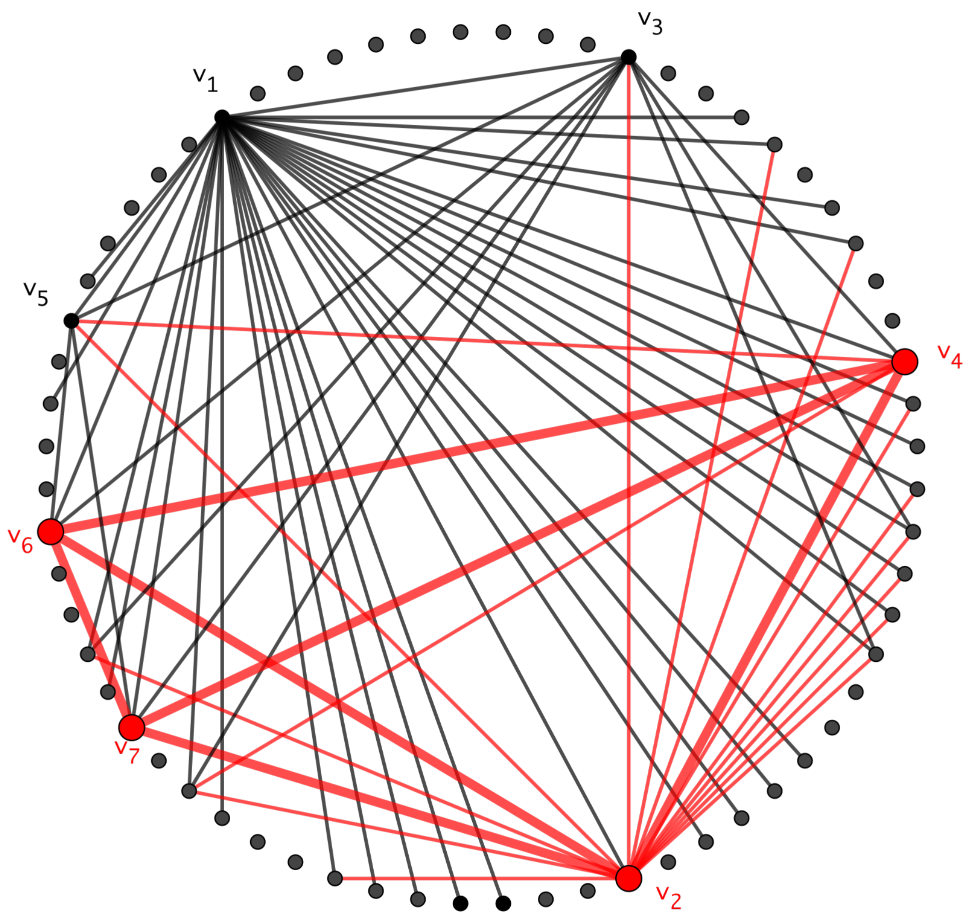
Step finale

Consideriamo quattro vertici dello stesso colore, diciamo v1, v3, v5, v7.
I vertici v3, v5 e v7 sono stati collegati a v1 in Step 1, quindi gli spigoli [v1,v3], [v1,v5] e [v1,v7] sono dello stesso colore v1.
I vertici v5 e v7 sono stati collegati a v3 in Step 3, quindi gli spigolo [v3,v5] e [v3,v7] sono dello stesso colore di v3 (e quindi di v1).
Il vertice v7 è stato collegato a v5 in Step 5, quindi lo spigolo [v5,v7] è dello stesso colore di v5 (e quindi di v1 e v3).
Quindi il grafo completo sui vertici v1, v3, v5, v7 è una clicca monocromatica di ordine 4.



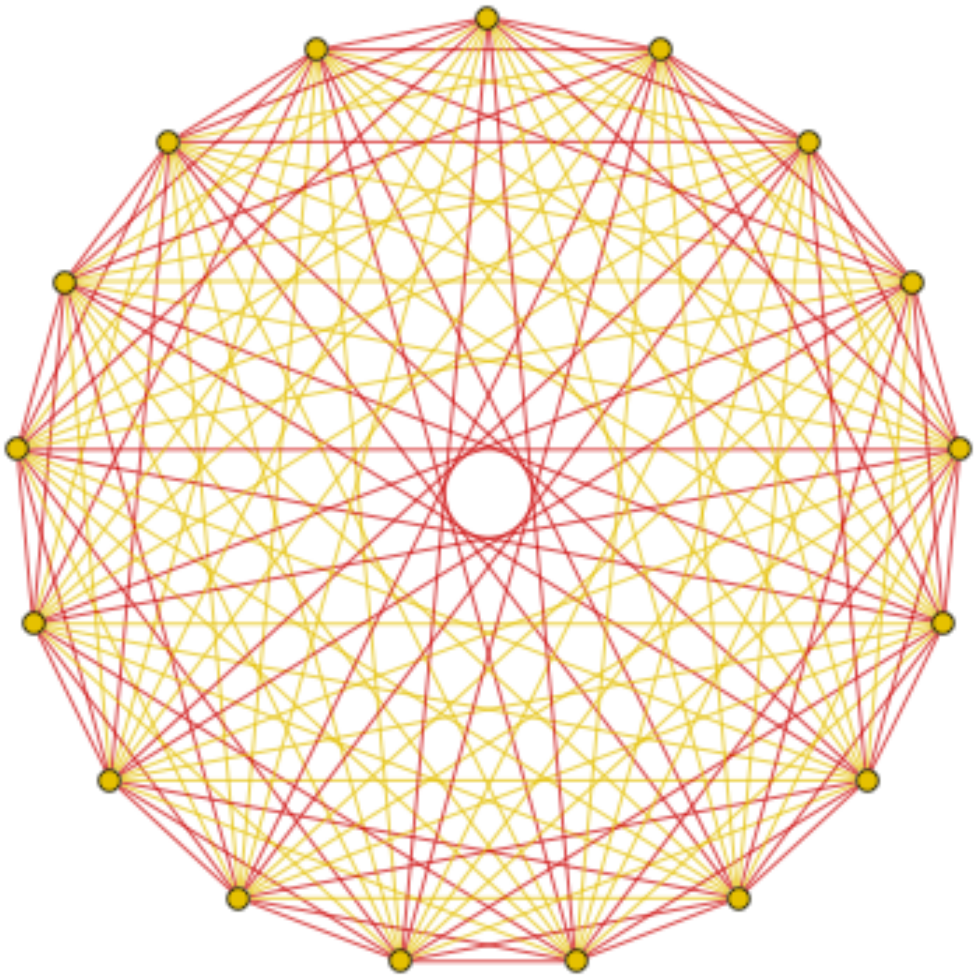
Esercizio

Come abbiamo detto, se avessi deciso, nell'esempio, di attribuire all'ultimo vertice il colore rosso, avresti avuto quattro vertici rossi: v2,v4,v6,v7. Verifica che questi quattro vertici, nell'esempio, definiscono una clicca rossa di ordine quattro.
Se avessi trovato 5 vertici dello stesso colore, invece che solo quattro, dimostra che, seguendo il procedimento indicato, avresti trovato nel grafo una clicca monocromatica di ordine 5.



Grafo giallo e rosso su 17 vertici privo di 4 clique

E’ possibile migliorare il bound precedente nel caso delle 4 clique, e dimostrare che già in ogni grafo con due colorazioni su 18 vertici esiste una 4-clique monocromatica.
Esistono invece un grafo su 17 vertici con due colorazioni privo di 4 clique monocromatiche.



Allora, per quanto visto sopra, il numero di vertici minimo per garantire 3 clique e 4 clique è rispettivamente R(3)=6, R(4)=18.

Importanza del teorema di Ramsey

Il teorema di Ramsey è un risultato fondazionale in combinatoria. La prima versione di questo risultato fu dimostrato da F. P. Ramsey e diede impulso alla teoria di Ramsey, che cerca la regolarità in mezzo al disordine, ovvero, definisce una una nozione di regolarità in una classe di strutture combinatorio e cerca di determinare condizioni per l'esistenza di strutture regolari entro strutture assegnate. Nel caso dei grafi con due colorazioni, la regolarità cercata è un sottografo completo monocromatico (clique).

I numeri di Ramsey

Il teorema di Ramsey afferma che, fissato k, esiste un N=N(k) abbastanza grande tale che, in ogni colorazione con due colori di un grafo completo su N vertici è possibile trovare una k clique monocromatica. Nella dimostrazione originale, Plumpton dimostrò il teorema assumendo $N(k) = 2^{2k}$. Molto lavoro nell'ambito di questo ramo della combinatoria è stato rivolto a trovare stime migliori del numero $N(k)$. Per ogni k, il minimo valore che rende vera la tesi di Ramsey, prende il nome di *Numero di Ramsey di ordine k*, indicato *R(k)*. Come spesso avviene in matematica, conviene considerare una situazione più generale da quella da cui si era partiti per poter collegare in maniera efficace ed iterativa i conteggi che si vogliono fare. Si è considerato quindi, per ogni coppia di interi r ed s, il problema di trovare in un grafo completo bicolorato diciamo in rosso e blu, che contenesse una r-clique rossa o una s-clique blu. Il più piccolo intero positivo *R(r,s)* tale che ogni grafico con *R(r,s)* vertici verifica la condizione richiesta di chiama numero di Ramsey di ordine *R(r,s).* Notiamo che *R(k)=R(k,k) e che R(r,s)=R(s,r)*..

Cosa sappiamo dei numeri di Ramsey?

s ***r***	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25	36–41	49–61	59–84	73–115	92–149
5					**43–48**	58–87	80–143	101–216	133–316	149–442
6						**102–165**	115–298	134–495	183–780	204–1171
7							**205–540**	217–1031	252–1713	292–2826
8								**282–1870**	329–3583	343–6090

9									**565–6588**	581–12677
10										**798–23556**

Gli alieni di Erdos.

Quanto è difficile calcolare esattamente i numeri di Ramsey?
Al grande matematico ungherese di origine ebraica Paul Erdős è attribuito il seguente aneddoto. Si immagini una forza aliena, dotata di armamenti molto più sofisticati e potenti di quelli terrestri, che invii una delegazione sulla terra che chieda di calcolare il valore esatto di *R*(5, 5) entro un anno, pena la distruzione del pianeta. Secondo Erdős ci converrebbe cooptare i più potenti calcolatori e i più bravi matematici del pianeta con la fondata speranza di venire a capo del problema. Se invece ci avessero chiesto di calcolare *R*(6, 6) allora ci converrebbe impiegare ogni risorsa per cercare di combattere gli alieni.
Morale sull'importanza delle applicazioni della matematica alla teoria (e alla pratica) delle decisioni.

Ci possiamo domandare perché sia così difficile calcolare esattamente un numero di Ramsey. Non possiamo usare il calcolatore? C'è solo un numero finito di 2-colorazioni di un grafo completo. Non possiamo ciclare sut tutte le colorazioni di un grafo, per esempio K(48) e vedere se contiene una 5 clique monocromatica? Se esiste una colorazione che non contiene una 5-clique monocromatica, allora, R(5) = 49. Se invece ogni colorazione contiene una 5 clique monocromatica, allora dobbiamo testare tutte le 2-colorazioni di K47, ecc., fino a determinare il quinto numero di Ramsey.

Il problema di questa strategia, basata sulla forza bruta, è che il numero delle colorazioni da considerare è estremamente elevato. Un K48 ha Bin(48,2)=1128 spigoli. Ogni spigolo si puo' colorare in due maniere e quindi dobbiamo controllare $2^{1128} \sim 3.6{\times}10^{339}$colorazioni. Al 2018, i più veloce supercomputer al mondo, il Cray Titan, poteva eseguire 20×10^15 floating-point operazioni al secondo (FLOPS). Se assumiamo (non realisticamente) che si possa verificare se una colorazione contiene una 5 clique monocromatica, con una singola operazione in virgola mobile, sarebbero necessari più di 10^315. anni per verificarle tutte. Sembra assai plausibile però che la terra venga assorbita dal sole in meno di 10^10 years.

Chi era Paul Erdos?

Paul Erdős (1913-1996) came from a Jewish family (the original family name being Engländer) although neither of his parents observed the Jewish religion. Paul's father Lajos and his mother Anna had two daughters, aged three and five, who died of scarlet fever just days before Paul was born. This naturally had the effect making Lajos and Anna extremely protective of Paul. He would be introduced to mathematics by his parents, themselves both teachers of mathematics.

Paul was not much over a year old when World War I broke out. Paul's father Lajos was captured by the Russian army as it attacked the Austro-Hungarian troops. He spent six years in captivity in Siberia. As soon as Lajos was captured, with Paul's mother Anna teaching during the day, a German governess was employed to look after Paul. Anna, excessively protective after the loss of her two daughters, kept Paul away from school for much of his early years and a tutor was provided to teach him at home.

The situation in Hungary was chaotic at the end of World War I. After a short while as a democratic republic, a communist Béla Kun took over, and Hungary became a left wing Soviet Republic. Anna was at this time made head teacher of her school but when the Communists called for strike action against Kun's regime she continued working, not for political reasons but simply because she did not wish to see children's education suffer.

After four months in control of Hungary, Kun fled to Vienna when Romanian troops advanced on Budapest in July 1919. Miklós Horthy, a right-wing nationalist, took over control of the country. He quickly moved against those perceived as Communists and Anna Erdős fell into that category due to her failing to obey the Communist strike call when Kun was in power. She was dismissed from her post and she was left in fear of her life as Horthy's men roamed the streets killing Jews and Communists. By 1920 Horthy had introduced anti-Jewish laws similar to those Hitler would introduce in Germany thirteen years later.

The year 1920 was not all bad for Paul, for his father Lajos returned home from Siberia. He had learnt English to pass the long hours in captivity but, having no English teacher, did not know how to pronounce the words. He now set about teaching Paul to speak English, but the strange English accent which this gave Paul remained one of his characteristics throughout his life.

Despite the restrictions on Jews entering universities in Hungary, Erdős, as the winner of a national examination, was allowed to enter in 1930. He studied for his doctorate at the University Pázmány Péter in Budapest. Awarded a doctorate in 1934, he took up a post-doctoral fellowship at Manchester, essentially being forced to leave Hungary because he was Jewish. During his tenure of the fellowship, Erdős travelled widely in the UK. He met Hardy in Cambridge in 1934 and Ulam, also in Cambridge, in 1935. His friendship with Ulam was to prove important later when Erdős was in the United States.

The situation in Hungary by the late 1930s clearly made it impossible for someone of Jewish origins to return. However he did visit Budapest three times a year during his tenure of the Manchester fellowship. In March 1938 Hitler took control of Austria and Erdős had to cancel his intended spring visit to Budapest. He did visit during the summer vacation but the Czech crisis on 3 September 1938 made him decide to return hurriedly to England. Within weeks Erdős was on his way to the USA where he took up a fellowship at Princeton. He hoped for his fellowship to be renewed but Erdős did not conform to Princeton's standards so he was offered only a six month extension rather than the expected year. Princeton found him [3]:-

> *... uncouth and unconventional...*

and Ulam invited Erdős to visit Madison to help out. We shall return later to give further details of the strange life which Erdős lived from this time on, devoted exclusively to seeking out and solving good mathematical problems. First we make some comments about his mathematics.

The contributions which Erdős made to mathematics were numerous and broad. However, basically Erdős was a solver of problems, not a builder of theories. The problems which attracted him most were problems in combinatorics, graph theory, and number theory. He did not just want to solve problems, however, he wanted to solve them in an elegant and elementary way. To Erdős the proof had to provide insight into why the result was true, not just provide a complicated sequence of steps which would constitute a formal proof yet somehow fail to provide any understanding.

Some results with which Erdős is most closely associated had been first proved before Erdős was born. In 1845 Bertrand conjectured that there was always at least one prime between n and $2n$ for $n \geq 2$. Chebyshev proved Bertrand's conjecture in 1850 but when Erdős was only an eighteen year old student in Budapest he found an elegant elementary proof of this result. Another result on prime numbers associated with Erdős is the Prime Number Theorem, namely:-

> *.. the number of primes* $\leq n$ *tends to* ∞ *as* $n/\log_e n$.

The theorem was conjectured in the 18^{th} century, Chebyshev himself came close to a proof, but it was not proved until 1896, when Hadamard and de la Vallée Poussin independently proved it using complex analysis. In 1949 Erdős and Atle Selberg found an elementary proof. Subsequent events are described in [15]:-

> *Selberg and Erdős agreed to publish their work in back-to-back papers in the same journal, explaining the work each had done and sharing the credit. But at the last minute Selberg ... raced ahead with his proof and published first. The following year Selberg won the Fields Medal for this work. Erdős was not much concerned with the competitive aspect of mathematics and was philosophical about the episode.*

This result was typical of the type of mathematics Erdős worked on. He posed and solved problems that were beautiful, simple to understand, but notoriously difficult to solve.

Erdős did receive the Cole Prize of the American Mathematical Society in 1951 for his many papers on the theory of numbers, and in particular for the paper *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem* published in the Proceedings of the National Academy of Sciences in 1949.

Whether a rather silly event which took place in August 1941 was to have any real effect on Erdős's life, or whether it was simply used as an excuse, is hard to tell. Erdős and two fellow mathematicians were picked up by the police near a military radio transmitter on Long Island. It was quite an innocent event with the three mathematicians being too absorbed in discussion of mathematics to notice a NO TRESPASSING sign. After a friendly session with the police it was realised that no harm had been intended. However, it gave Erdős an FBI record which was later used against him.

Ulam left Madison in 1943 to join other mathematicians and physicists at Los Alamos in New Mexico working on the atomic bomb project. He asked Erdős to join the project but, although he was interested enough to be interviewed, Erdős gave answers to those interviewing him which he must have known were not what they wanted to hear. Erdős was simply too honest in saying that he would wish to return to Budapest at the end of the war. This episode does give the feeling that Erdős never wanted to work at Los Alamos, but was simply amusing himself.

In 1943 Erdős worked at Purdue University, taking a part-time appointment. Although it was a difficult time with great uncertainty about the fate of his family in Hungary, yet mathematically Erdős flourished. He had heard nothing from his family between 1941 and the time when Budapest was liberated in 1945. The Jews in Hungary had suffered incredible hardship from 1944 with many being murdered, and others deported to Auschwitz. It is unlikely that the full extent of the horror was understood by Erdős in the United States at the time. However, in August 1945, Erdős received a telegram giving details of his family. His father had died of a heart attack in 1942. His mother had survived while, quite remarkably, a cousin Magda Fredro had been sent to Auschwitz but had survived. The family had suffered terribly through the Nazi campaign against the Jews, however, and four of Erdős's uncles and aunts had been murdered.

Near the end of 1948 Erdős was able to return to Hungary for a visit and there he was reunited with his surviving family and friends. For the next three years he travelled frequently between England and the United States before accepting a temporary post at the University of Notre Dame in 1952. It was an inspired offer which gave Erdős complete freedom to rush off to do some joint research whenever he wanted. Erdős could not bring himself to accept the same generous offer on a permanent basis, which both the University of Notre Dame and Erdős's friends tried hard to encourage him to accept.

During the early 1950s senator Joseph R McCarthy whipped up strong feelings against communism in the United States. Erdős began to come under suspicion from authorities who saw imaginary problems everywhere. When asked by US immigration, as he returned after a conference in Amsterdam in 1954, what he thought of Marx, Erdős made the ill judged reply:-

> *I'm not competent to judge, but no doubt he was a great man.*

This was followed by a line of questioning about whether he would ever return to Hungary. Erdős said:-

> *I'm not planning to visit Hungary now because I don't know whether they would let me back out. I'm planning to go only to England and Holland.*

So, was it only the fear of not being let out of Hungary that stopped him going there. Erdős replied innocently:-

> *Of course, my mother is there and I have many friends there.*

Erdős was not allowed back to the United States but no reason was given. The files indicate that the official reasons were not the answers Erdős gave to the above questions, but the fact that he had corresponded with a Chinese mathematician who had subsequently returned from the United States to China and also Erdős's 1941 FBI record.

He spent much of the next ten years in Israel. During the early 1960s he made numerous requests to be allowed to return to the United States and a visa was finally granted in November 1963. By this time, however, Erdős had become a traveller moving from one university to another, and from the home of one mathematician to another. However, he did have a home of sorts with his friend Ronald Graham. Erdős and Graham met at a number theory conference in 1963 and soon began a mathematical collaboration. It was Graham who provided a room in his house where Erdős could live when he wanted, he also stored Erdős's papers there and, in many ways, acted as a secretary to Erdős.

Although somewhat over the top, the following quote from [12] shows the high regard in which Erdős was held by his fellow mathematicians:-

> *Never, mathematicians say, has there been an individual like Paul Erdős. He was one of the century's greatest mathematicians, who posed and solved thorny problems in number theory and other areas and founded the field of discrete mathematics, which is the foundation of computer science. He was also one of the most prolific mathematicians in history, with more than* 1,500 *papers to his name. And, his friends say, he was also one of the most unusual.*

Erdős won many prizes including the Wolf Prize of 50 000 dollars in 1983. However he had a lifestyle that needed little money and he gave away:-

> *... most of the money he earned from lecturing at mathematics conferences, donating it to help students or as prizes for solving problems he had posed.*

In 1976 Ulam gave this description of Erdős:-

> *He had been a true child prodigy, publishing his first results at the age of eighteen in number theory and in combinatorial analysis. Being Jewish he had*

to leave Hungary, and as it turned out, this saved his life. In 1941 *he was twenty-seven years old, homesick, unhappy, and constantly worried about the fate of his mother who remained in Hungary. ... Erdős is somewhat below medium height, an extremely nervous and agitated person. ... His eyes indicated he was always thinking about mathematics, a process interrupted only by his rather pessimistic statements on world affairs, politics, or human affairs in general, which he viewed darkly. ... His peculiarities are so numerous it is impossible to describe them all. ... Now over sixty, he has more than seven hundred papers to his credit.*

Successioni policromatiche di interi e successioni monocromatiche regolari

Colorazioni ordinate: quali cercare? Disegnare una successione monocromatica "ordinata".

10 caselle colorate

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

5 caselle colorate

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

7 caselle colorate

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Progressioni aritmetiche.

Definire una successione monocromatica “ordinata”.
Definire progressioni aritmetiche, trigressioni, 4 gressioni, ecc.
Definire configurazioni ordinate in una tabella.

Successioni binarie.

Come si riconoscono in una successione policromatica Esercizio.
Algoritmo per riconoscere la più lunga progressione aritmetica policromatica di interi

Gioco di Van der Waerden

Gioco: A comincia, colorando una casella di rosso.
B risponde, colorando una casella di blu. Si ripete lo schema fino a che, A vince, se riesce a colorare una tri-gressione. Altrimenti vince B.

1	2	3	4	5	6	7	8

A vince, se riesce a colorare una tri-gressione. Altrimenti vince B.

1	2	3	4	5	6	7	8	9

A vince, se riesce a colorare una 4-gressione. Altrimenti vince B.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

A vince, se riesce a colorare una 4-gressione. Altrimenti vince B.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36

Analogie con il gioco di Ramsey? Problematiche analoghe?

Teorema di Van der Waerden

Per r (numero dei colori) e k (lunghezza della progressione) esiste un intero N(r,k) tale che in sequenza policromatica di interi colorata con r colori e di lunghezza N(r,k) esiste una k progressione monocromatica.

Chi era Bartel Van der Waerden?

Bartel van der Waerden (1903-1996)'s parents were Theodorus van der Waerden and Dorothea Adriana Endt. Theo's father, Bartel's paternal grandfather, was Hendricus Johannes van der Waerden who owned a large blacksmith business. Theo was born in Eindhoven on 21 August 1876 and studied civil engineering at the Delft Technical University. Then, after teaching mathematics and mechanics in Leeuwarden and Dordrecht, he moved to Amsterdam in 1902 where again he taught mathematics and mechanics. At university he had become interested in politics and played a role in politics throughout his life as a left wing Socialist. He married Dorothea on 28 August 1901. Her parents were Coenraad Endt and Maria Anna Kleij who were Dutch Protestants. Bartel was the eldest of their three children, the other two boys being Coenraad (born 29 December 1904) and Benno (born 2 October 1909). In 1911 Theo was awarded the degree of Doctor of Technical Sciences, and was elected as a member of the SDAP (Sociaal-Democratische Arbeiderspartij) to the Provincial government of North Holland in 1910.

As a child, van der Waerden was not allowed to read his father's mathematics books but was told to play outside. This made him fascinated to discover mathematics for himself. After elementary school, van der Waerden entered the Hogere Burger School of Amsterdam in 1914. As a school pupil at the Hogere Burger School, van der Waerden showed remarkable promise and he developed for himself the laws of trigonometry. He studied mathematics at the University of Amsterdam, beginning his course in 1919 at the age of sixteen. He learnt topology from Gerrit Mannoury who was a friend of his father (Mannoury was a Communist and Theo, although not a Communist himself, had many friends in that party). He also learnt invariant theory from Roland Weitzenböck. Dirk van Dalen [3] paints a picture of van der Waerden as an undergraduate:-

> *The study of mathematics was for him the proverbial 'piece of cake'. Reminiscing about his studies he said: "I heard Brouwer's lectures, together with Max Euweand Lucas Smidt. The three of us listened to the lectures, which were very difficult." ... Van der Waerden meticulously took notes in class, and usually*

that was enough to master all the material. Brouwer's class was an exception. Van der Waerden recalled that at night he actually had to think over the material for half an hour and then he had in the end understood it.

Van der Waerden was an extremely bright student, and he was well aware of this fact. He made his presence in class known through bright and sometimes irreverent remarks. Being quick and sharp (much more so than most of his professors) he could make life miserable for the poor teachers in front of the blackboard. During the, rather mediocre, lectures of Van der Waals Jr he could suddenly, with his characteristic stutter, call out: "Professor, what kind of nonsense are you writing down now?" He did not pull such tricks during Brouwer's lectures, but he was one of the few who dared to ask questions.

Brouwer however, did not like to be questioned and his assistant spoke to van der Waerden asking him to ask no further questions during lectures. After taking his first degree in Amsterdam he went to Göttingen for seven months to study under Emmy Noether. Brouwer wrote to Hellmuth Kneser at Göttingen on 21 October 1924 before van der Waerden went there (see [3]):-

In some days my student (or actually Weitzenböck's) will come to Göttingen for the winter semester. His name is Van der Waerden, he is very clever and has already published (namely in Invariant Theory).

At Göttingen, van der Waerden learnt much topology from Hellmuth Kneser. He said [7]:-

... from the beginning I was in contact with him, and from him I really learned topology. Kneser and I used to have lunch together; after having eaten he went home, but on occasion we first took a brief walk. We strolled through the woods of Göttingen, and he taught me many things.

Dirk van Dalen writes [3]:-

Once in Göttingen under Emmy's wings, Van der Waerden became a leading algebraist. Emmy was very pleased with the young Dutchman, "That Van der Waerden would give us much pleasure was correctly foreseen by you. The paper he submitted in August to the Annalen is most excellent (Zeros of polynomial ideals) ... ", she wrote to Brouwer on 14 November 1925.

Van der Waerden returned to the Netherlands in 1925 where he both wrote his doctoral dissertation, supervised by Hendrik de Vries, and undertook military duty at the marine base in Den Helder. Dirk van Dalen writes [3]:-

In mathematics Van der Waerden was easily recognised as an outstanding scholar, but in the 'real world' he apparently did not make such a strong impression. When Van der Waerden spent his period of military service at the naval base in Den Helder, a town at the northern tip of North-Holland, his Ph.D. advisor [Hendrik de Vries] visited him one day. He said that the commander was not impressed by the young man, "he is a nice guy but not very bright."

His doctoral thesis *De algebraiese grondslagen der meetkunde van het aantal* Ⓣ was submitted to the University of Amsterdam and he defended it in the grand hall of the University on 24 March 1926. He had been awarded a Rockefeller fellowship for a year and, following the semester in Göttingen with Emmy Noether, he went to Hamburg to study for a semester with Hecke, Artin and Schreier. There he attended Artin's algebra course and took notes with the aim of writing a joint book with him. However, when later Artin saw the part of the text van der Waerden was writing, he suggested that he write the whole book without any chapters being contributed by Artin. This eventually became van der Waerden's famous text *Moderne Algebra*. The year 1927 was a busy one for van der Waerden. He was offered a position at the University of Rostock but was appointed to a lectureship at Groningen in the same year. He returned to Göttingen as a visiting professor in 1929 and in July of that year he met Camilla Rellich, sister of the Franz Rellich who was completing his doctoral thesis under Richard Courant. Van der Waerden married Camilla in September 1929 and the two returned to Groningen. There he continued working on *Moderne Algebra* which contained much material from Emmy Noether's lectures as well as those of Artin. Volume I was published in 1930 while volume II, which contains much of van der Waerden's own work, was published in the following year.

In 1931 he was appointed professor of mathematics at the University of Leipzig where he became a colleague of Werner Heisenberg. His interaction with Heisenberg and other theoretical physicists led to van der Waerden publishing *Die gruppentheoretische Methode in der Quantenmechanik* in 1932. He then began to publish a series of articles in *Mathematische Annalen* on algebraic geometry. In these articles, van der Waerden defined precisely the notions of dimension of an algebraic variety, a concept intuitively defined before. His work in algebraic geometry uses the ideal theory in polynomial rings created by Artin, Hilbert and Emmy Noether. His work also makes considerable use of the algebraic theory of fields. However, he later changed his approach as is evident in his book *Einführung in die Algebraische Geometrie* (1939). Dan Pedoe writes in a review:-

> *About ten years ago, van der Waerden, already eminent as an algebraist, began, in a series of papers in the Mathematische Annalen, to create rigorous foundations for algebraic geometry. The implication-that there was something unsound in the magnificent structure of Italian geometry-was vigorously contested by Severi. Fortunately, van der Waerden continued his researches, but with the implicit sub-title, "An algebraist looks at algebraic geometry". With increasing knowledge of the powerful methods of the Italian school, he has gladly modified his own methods. Ideal-theory, the weapon of attack in his first papers, he has found almost completely unnecessary ... As a result of the experience gained in writing these papers, and in giving various courses of lectures, Professor van der Waerden has produced a work which must sooner or later find a place on every geometer's bookshelves.*

In 1934 van der Waerden joined the main editorial board of *Mathematische Annalen*. This was a difficult time to take on such a role since he came under pressure from the Nazis not to publish papers by Jewish authors. This pressure made him think about resigning, but when he was informed that if he did so it was likely that one of the Nazis Wilhelm Blaschke or Ludwig Bieberbach would replace him, he decided to continue. Both before and after the start of World War II, van der Waerden, as a foreigner, had problems from the Nazis. Although working in Germany he refused to give up his Dutch citizenship and his life was made difficult. He wrote on 16 May 1940 (see [1]):-

> *In itself I have nothing against German citizenship, however, at this moment, since Germany has occupied my homeland, I would not gladly give up my previous neutrality and throw myself in a certain measure publicly on the German side.*

This letter was written one day after the German invasion of the Netherlands was complete. At this time his parents were still in the Netherlands. They had moved from Amsterdam to Laren near the end of the 1920s after their children had left home. Theo had built a fine home there but shortly after the German invasion, on 12 June 1940, he died of cancer. Dorothea continued to live there with her daughter but she was so distressed by the German occupation that she committed suicide on 14 November 1942, drowning herself in a lake near her home. On 4 December 1943 the van der Waerden home in Leipzig was bombed and Bartel and Camilla van der Waerden, together with their three children Helga, Ilse and Hans, left for Dresden where Camilla's brother Franz Rellich was professor of mathematics. Here the situation was equally bad so they accepted an invitation from one of van der Waerden's students to live with her in Bischofswerda, a small town near Dresden. They remained there for nearly a year before returning to Leipzig. The city was under continual air attack and in 1945, unable to take the strain any longer, they moved to Austria to live in the country at Tauplitz, near Graz, with Camilla's mother. In July 1945 American soldiers arrived in Tauplitz and told everyone to return to their country of origin. The van der Waerdens returned to the Netherlands and lived in the house Theo van der Waerden had built in Laren.

Van der Waerden now had no job and hardly any money to buy food for his family. He was offered a post at Utrecht University, arranged by Hans Freudenthal, but because he had worked throughout the war in Germany, the government refuse to allow him to take up the position. Freudenthal then managed to obtain a position for van der Waerden working for Shell in Amsterdam on applied mathematics. In 1947 he visited the United States, going to Johns Hopkins University where he was offered a permanent post. He refused the offer and returned in 1948 to a chair of mathematics at the University of Amsterdam where he remained until 1951. In 1950 Karl Fueter died and van der Waerden was appointed to fill the vacant chair in Zürich in 1951. His impact on the department in Zürich was very great. As well as an almost unbelievable range of mathematical research interests, van der Waerden stimulated research in Zürich by supervising over 40 doctoral students during his years there. In fact van der Waerden was to remain in Zürich for the rest of his life.

Van der Waerden worked on algebraic geometry, abstract algebra, groups, topology, number theory, geometry, combinatorics, analysis, probability theory, mathematical statistics, quantum mechanics, the history of mathematics, the history of modern physics, the history of astronomy and the history of ancient science. We have already mentioned Van der Waerden's most famous book, *Moderne Algebra* published in 1930-1931. In Galois theory he showed the asymptotic result that almost all integral algebraic equations have the full symmetric group as Galois group. He produced results in invariant theory, linear groups, Lie groups and generalised some of Emmy Noether's results on rings. In group theory he studied the Burnside groups $B(3, r)$ with r generators and exponent 3. These are solutions of the Burnside problem. These groups were shown to be finite by Burnside. In 1933 van der Waerden found the exact order and structure of the groups $B(3, r)$. He showed that the order of $B(3, r)$ is $3^{N(r)}$ where the exponent

$$N(r) = r + r(r - 1)/2 + r(r - 1)(r - 2)/6.$$

Among his many historical books are *Ontwakende wetenschap* Ⓣ (1950) translated into English as *Science Awakening* (1954), *Science Awakening II: The Birth of Astronomy*(1974), *Geometry and Algebra in Ancient Civilizations* (1983), and *A History of Algebra* (1985). Dirk Struik, reviewing the first of these, writes:-

> *This is the first book which bases a full discussion of Greek mathematics on a solid discussion of pre-Greek mathematics. Carefully using the best sources available at present, the author acquaints the reader not only with the work of Neugebauer and Heath, but also with that of the philological critics who centered around the "Quellen und Studien." ... This book contains a wealth of material, critically arranged, and reads exceedingly well. It has an original approach and contains much novel material.*

As to *A History of Algebra*, Jeremy Gray writes in a review:-

> *It is almost unfailingly clear. The arguments presented are summarized with a deftness that isolates and illuminates the main points, and as a result they are frequently exciting. Since nearly* 200 *pages of it are given over to modern developments which are only now receiving the attention of historians, this book should earn itself a place as an invaluable guide. Its second virtue is the zeal with which the author has attended to the current literature. Almost every section gives readers an indication of0 where they can go for a further discussion. As a result, many pieces of information are here presented in book form that might otherwise have languished in the scholarly journals. Since one must be cynical of the mathematicians' awareness of those journals, the breadth and generosity of van der Waerden's scholarship will do everyone a favour.*

The history of mathematics was not a topic he just turned to late in life. He explained [7]:-

> *When I was a student, Hendrik de Vries gave a course on the history of mathematics. After that I read Euclid and some of Archimedes. Thus, my interest began very early. At Göttingen - the first time I was there - I attended the lectures of Neugebauer, who gave a course on Greek mathematics.*

Van der Waerden's important paper *Die Arithmetik der Pythagoreer* Ⓣ appeared in 1947 followed by *Die Astronomie der Pythagoreer* Ⓣ in 1951.

In 1973 van der Waerden retired from his chair in Zürich. He continued to undertake research in the history of mathematics publishing around 60 papers after he retired. The papers which appeared in the years 1986-88 include: *Francesco Severi and the foundations of algebraic geometry* (1986), *On Greek and Hindu trigonometry* (1987), *The heliocentric system in Greek, Persian and Hindu astronomy* (1987), *The astronomical system of the Persian tables* (1988), On the *Romaka-Siddhanta* (1988), *Reconstruction of a Greek table of chords* (1988), and *The motion of Venus in Greek, Egyptian and Indian texts* (1988). Although several of his publications appeared after 1988, all were taken from lectures he had given earlier.

I numeri di Van der Waerden

Il teorema di Van der Waerden afferma che, per ogni coppia di interi positivi *r* and *k* esiste un intero positiv *N*=N(r,k) tale che, se gli interi {1, 2, ..., *N*} sono colorati, ognuno con uno a scelta tra *r* colori diversi, allora esistono almeno *k* interi in progressione aritmetica, ognuno de quali ha lo stesso colore. Il più piccolo di questi *N* è il **numero di van der Waerden** *W*(*r*, *k*).

Cosa sappiamo dei numeri di Van der Waerden

k\r	2 colors	3 colors	4 colors	5 colors
3	9	27	76	>170
4	35	293	>1,048	>2,254
5	178	>2,173	>17,705	>98,740
6	1,132	>11,191	>91,331	>540,025
7	>3,703	>48,811	>420,217	>1,381,687
8	>11,495	>238,400	>2,388,317	>10,743,258
9	>41,265	>932,745	>10,898,729	>79,706,009
10	>103,474	>4,173,724	>76,049,218	>542,694,970
11	>193,941	>18,603,731	>305,513,570	>2,967,283,511

Dimostrazione del teorema di Van der Waerden

Applicazioni del teorema di Van der Waerden

Teorema di Green - Tao. Esistono progressioni aritmetiche di numeri primi di lunghezza arbitraria. Non si tratta di una conseguebza ma di un sostanziale raffinamento.