Navier-Stokes equation:
how relevant the existence-uniqueness problem?

Abstract: Existence-uniqueness theorems may be too strict requirements for many problems in Physics: Statistical Mechanics flourishes studying systems for which no existence-uniqueness is available for most infinite systems to which ideally it should apply in studying thermodynamics. Here an analogy is proposed between the theory of the thermodynamic limit and the problem of fluids and turbulence discussing pro-and-con for a statistical interpretation of viscosity and reversibility of fluid motion, also with attention to recent computer simulations.

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Navier-Stokes equation: how relevant the existence-uniqueness problem?

Properties of a stationary state of an incompressible fluid in a periodic box of side $L = 2\pi$ and subject to a 'large scale' force f is considered.

'widely accepted', [1]: $\ell_K = LR^{-\frac{3}{4}}$ gives order of length-scale below which energy, input at large scale, is transferred to be dissipated by the viscosity action.

A stationary NS-states theory should predict at least averages of observables $O(\mathbf{u})$ depending on u via large scale components $\mathbf{u_k}$, *i.e.* Fourier's components $\mathbf{u_k}$ with $|\mathbf{k}| < c\,\ell_K^{-1}$, for some $c = O(1)$.

Represent a velocity $\mathbf{u}(\mathbf{x})$ represented as:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k},c=1,2} u^c_{\mathbf{k}}\, i\, \mathbf{e^c}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}},$$

with $||\mathbf{e^c}(\mathbf{k})|| = \mathbf{1}$, $\mathbf{e^c}(\mathbf{k}) = -\mathbf{e^c}(-\mathbf{k})$, $\mathbf{k}\cdot\mathbf{e}^c(\mathbf{k}) = 0$, $\mathbf{k} \neq \mathbf{0}$, $\overline{u}^c_{\mathbf{k}} = u^c_{-\mathbf{k}}$, $\mathbf{e^1}(\mathbf{k})\cdot\mathbf{e^2}(\mathbf{k}) = 0$. The equations of motion:

$$\dot{\mathbf{u}}(\mathbf{x}) = -(\underline{\mathbf{u}}(\mathbf{x})\cdot\boldsymbol{\partial})\mathbf{u}(\mathbf{x}) - \nu\Delta\mathbf{u}(\mathbf{x}) - \boldsymbol{\partial} p(\mathbf{x}) + \mathbf{f}(\mathbf{x})$$

$$\boldsymbol{\partial}\cdot\mathbf{u}(\mathbf{x}) = 0$$

and $\mathbf{f}(\mathbf{x}) = \sum_{0<|\mathbf{k}|\le k_{\max}} f^c_{\mathbf{k}}\, i\, \mathbf{e^c}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$: constraint $|\mathbf{k}| \le k_{\max}$ indicates that forcing occurs a large scale.

If NS is considered fundamentally correct, the prediction should concern all local observables, *i.e.* **all O's depending on finitely many harmonics:** *i.e.* **"large scale" or**

"LOCAL OBSERVABLES"

Unavoidable difficulty: no guarantee for a NS-solution, $S_t\mathbf{u}_0 \stackrel{def}{=} \mathbf{u}_t$, initiating at a smooth $\mathbf{u}_0$: *i.e.* **no stable algorithm exists for constructing $\mathbf{u}_t$**, see [2].

Research mostly devoted to *regularized* **NS eq.:** *i.e.* **modified so that** *a priori* **$\mathbf{u}(\mathbf{x})$ evolves remaining smooth and admits an algorithm to construct $(S_t\mathbf{u})(\mathbf{x}) = \mathbf{u}_t(\mathbf{x})$**

Example of regularization: let the NS as equations for the harmonics $u^c_{\mathbf{k}}$, *i.e.*

$$\dot{u}^c_{\mathbf{k}} = -\sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ a,b=1,2}} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u^a_{\mathbf{k}_1} u^b_{\mathbf{k}_2} - \nu \mathbf{k}^2 u^c_{\mathbf{k}} + f^c_{\mathbf{k}}$$

where $T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} \stackrel{def}{=} (e^a(\mathbf{k}_1)\cdot\mathbf{k}_2)(e^b(\mathbf{k}_2)\cdot e^c(\mathbf{k}))$.

Set $=0$ **all** $u^r_{\mathbf{h}}$ **with** $\mathbf{h}=(h_1,h_2,h_3)$ **and** $\max_i |h_i| > N$**.**

This ODE, named $\mathrm{INS}^{\mathrm{N}}$ **(Irreversible NS), is on a** $D_N = 2((2N+1)^3-1)$**-dimensional phase space, f is fixed once and for all (with** $||\mathbf{f}||_2 = 1$**, say, and on large scale (**$\mathbf{f}_{\mathbf{k}} = 0, |\mathbf{k}| > k_{max}$**)): depends on** $\boldsymbol{\nu}$**, only**

A general property of ODE's generating "chaotic motions", like the $\mathrm{INS}^{\mathrm{N}}$**, is immediately very important: it is to admit unique SRB-distributions** μ^N_ν **(or finitely many in case of more attractors).**

This means (Ruelle) that, aside from a zero volume set of data u, the averages of *all observables* O **are** $\boldsymbol{\mu}^N_{\boldsymbol{\nu}}(O)$ **and hence the stationary properties of the** $\mathrm{INS}^{\mathrm{N}}$ **evolution are completely determined by SRB or have a finite number of possibilities,** [3, 4, 5].

At $\boldsymbol{\nu}$**'s where uniqueness of the SRB distr. can be assumed, this solves the question "which is the probability distr. relevant for the averages" among the uncountably many stationary ones? answer the SRB distr.** $\boldsymbol{\mu}^N_{\boldsymbol{\nu}}$**.**

Of course the basic existence problem has not disappeared: forgetting that $\boldsymbol{\mu_\nu^N}$ is not known, the interest, if the NS equations are taken as fundamental, is entirely resting on the limits as $N \to \infty$ of the local observables averages.

However a similitude with Statistical Mechanics (SM) becomes manifest.

The cut-off N can be seen 'corresponding' to the volume V enclosing the rV hard core particles (say) of a gas of density r without other external forces.

Its (Hamiltonian) eqs. of motion are ODE's that can be seen as a regularization of the eqs. that would control motion of an ∞ gas (of density r, filling the Universe !) for which no constructive existence/uniqueness is known.

Still SM fared very well in absence of existence-uniqueness results for the evolution of the ∞-system, because of the physicists' attitude.

The SRB distribution can be related to the ergodic hypothesis and:

(1) find, or select, for finite V, a family of stationary distrib. $\boldsymbol{\mu}^V$, and use them to define by $\boldsymbol{\mu}^V(O)$ the averages of physically interesting observ.s ($i.e.$ the "local" observ.s O, whose value depends only on positions-velocities of particles located in a V*-independent region* inside the confining V, [6]).

(2) show $\lim_{V\to\infty}\boldsymbol{\mu}^V(O)=\langle\, O\,\rangle$ to exist $\forall$ "local" O,

(3) exhibit constraints between the average values

Item (1) is easy in SM *if the ergodic hypothesis is accepted*: **it allows restricting consideration to stationary $\boldsymbol{\mu}^V$'s uniform on energy surfaces,** [7].

Item (2) has been at the center of the study of the "thermodynamic limit", leading to the proofs that, in a very large number of models, the limit exists $\forall$ local observables, [6, 8].

Item (3) led to the great achievement of showing, in important models, that varying the systems parameters the averages invariably change in agreement with the variations *forseen by the laws of thermodynamics*, **provided V is large enough, [6, 9].**

Analogy with fluids

(1) Unif. distr. on energy surface can be regarded as a SRB distribution: correspondingly in $\mathrm{INS}^{\mathrm{N}}$ **(chaotic) the SRB is uniquely selected to describe the statistical properties. Solving the major problem of identifying the distr. for the statistical prop.s of a flow (exceptions allowed as in SM).**

The assumption is inherited from the microscopic motion of the fluid molecules, *even* **when the fluid flow is periodic (***e.g.* **if viscosity is large) as the fluid equations are derived via scaling limits, without change of the eq. of motion.**

(2)&(3) this is part of the existence/uniqueness for NS: open

The just sorted analogy leads to define: *viscosity ensemble* $\equiv$ **collection for** $\boldsymbol{\nu} > 0$ **of SRB stationary distributions** $\boldsymbol{\mu}_{\boldsymbol{\nu}}^{N}$ **for the** $\mathrm{INS}^{\mathrm{N}}$ **equation.**

For each $\boldsymbol{\nu} > 0$ **the distr.** $\boldsymbol{\mu}_{\boldsymbol{\nu}}^{N}$ **assigns the average** $\boldsymbol{\mu}_{\boldsymbol{\nu}}^{N}(O) = \langle\, O \,\rangle_{\nu}^{N}$ **of any local observ.** O **on a flow with initial data randomly selected with a distrib. with a continuous density** $\delta(\mathbf{u})$ **with respect to the volume in the** D_N**-dimensional phase space.**

In the corresponding SM case the microcanonical distribution $\boldsymbol{\mu}_{E}^{V}(d\mathbf{p}d\mathbf{q})$**, for a system of particles of total energy** E **enclosed in a volume** V**, assigns the average value** $\boldsymbol{\mu}_{E}^{V}(O) = \langle\, O \,\rangle_{E}^{V}$ **to any local observable.**

At this point we ask whether it is possible to define other collections $\mathcal{E}^N$ of stationary distributions $\boldsymbol{\lambda}_{\boldsymbol{\gamma}}^N$ which, depending on a parameter $\boldsymbol{\gamma}$, will assign averages $\langle O \rangle_{\gamma}^N =$ so that a correspondence $\boldsymbol{\nu} \longleftrightarrow \boldsymbol{\gamma}$ can be established in the form $\boldsymbol{\gamma} = g_N(\boldsymbol{\nu})$ implying:

$$\lim_{\mathbf{N}\to\infty} \mu_{\boldsymbol{\nu}}^{\mathbf{N}}(\mathbf{O}) = \lim_{\mathbf{N}\to\infty} \boldsymbol{\lambda}_{\boldsymbol{\gamma}}^{\mathbf{N}}(\mathbf{O}) \qquad \text{if} \quad \boldsymbol{\gamma} = g_N(\boldsymbol{\nu})$$

Then we shall say that the ensembles $\mathcal{E}_{viscosity}^N$ and $\mathcal{E}^N$ are *equivalent* (in the $N \to \infty$ limit).

Just as we call microcanonical distr.s $\boldsymbol{\mu}_E^V$ equivalent to canonical ones $\boldsymbol{\lambda}_{\boldsymbol{\beta}}^V$ in the limit as $V \to \infty$ provided β and E are suitably related. [8].

Viscosity phenomenologically describes an average over chaotic microscopic motions it is conceivable that it could be replaced by another force subject to rapid fluctuations with average $\boldsymbol{\nu}$, while properties of large scale observables (*i.e.* the local ones) will be neglegibly affected.

Describing the same system with different equations which become equivalent for practical purposes (and even rigorously in suitable limits) for a vast class of observables is familiar in SM: an example is the equivalence between the microcanonical and the isokinetic ensembles.

In a system of $\mathcal{N} = rV$ mass $m = 1$ particles in a cubic vessel V, interacting, with the walls and reciprocally, via repulsive short range potential $\varphi(\mathbf{q})$ consider 2 equations:

$$\dot{\mathbf{q}} = \mathbf{p}, \;\; \dot{\mathbf{p}} = -\boldsymbol{\partial}_{\mathbf{q}}\varphi(\mathbf{q}) \;\; -\boldsymbol{\alpha}(\mathbf{q},\mathbf{p})\mathbf{p}$$

where the multiplier $\boldsymbol{\alpha}$ is $= 0$ (Hamiltonian) or so defined that the second equation admits the total kinetic energy $\frac{1}{2}\mathbf{p}^2$ *exactly* constant (Isokinetic).

A brief calculation yields the isokinetic value of $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}(\mathbf{q},\mathbf{p}) = -\frac{\mathbf{p}\cdot\boldsymbol{\partial}_{\mathbf{q}}\varphi(\mathbf{q})}{\mathbf{p}^2}$$

Stationary distr.s of the first are microcanonical

$$\mu_{\varepsilon}^{V}(d\mathbf{p}d\mathbf{q}) = \frac{1}{Z_{\varepsilon,V}}\ \delta(\frac{\mathbf{p}^2}{2} + \varphi(\mathbf{q}) - \varepsilon\mathcal{N})\ d\mathbf{p}d\mathbf{q}$$

while stationary distributions for the second are:

$$\lambda_{\beta}^{V}(d\mathbf{p}d\mathbf{q}) = \frac{1}{Z_{\beta,V}}\ \delta(\frac{\mathbf{p}^2}{2} - \frac{3}{2}\beta^{-1}\mathcal{N})\ d\mathbf{p}d\mathbf{q}$$

(by direct check, [10]). In absence of phase trans., often local observables O verify

$$\lim_{V\to\infty}\mu_{\varepsilon}^{V}(O) = \lim_{V\to\infty}\lambda_{\beta}^{V}(O)$$

under the "equivalence condition"

$$\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}^{\mathbf{V}}(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{p}^{\mathbf{2}}) = \frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{\beta}^{-\mathbf{1}}\mathcal{N}$$

Since 1980's different equations are used describing the same system and yielding same averages to interesting observables (at least approximately: complete equivalence could only be in limit situations, like $V \to \infty$, not really accessible).

Vast literature on simulations on nonequilibrium, [10, 11, 12] provides many examples.

Different equations for the same system: usually obtained by adding to the equations new forces so designed to turn one, or more, selected (typically non-local) observable into a constant of motion.

The extra forces have been often interpreted as simulating the action of "thermostats": such are the "Nosè-Hoover" thermostats, [13], or the "Gaussian" thermostats, [14]. It is even possible to impose simultaneously many extra forces: a most remarkable case in [15] concerns the NS eq..

Selection of the observables which, via the modification of equations, must remain constant is addressed towards quantities that are expected to have small fluctuations in a limit situation of interest, like kinetic energy in the example.

Coming back to the NS equations

$$\dot{u}^c_{\mathbf{k}} = - \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k} \\ a,b=1,2}} T^{a,b,c}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}} u^a_{\mathbf{k}_1} u^b_{\mathbf{k}_2} - \boldsymbol{\nu} \mathbf{k}^2 u^c_{\mathbf{k}} + f^c_{\mathbf{k}} \qquad (\#)$$

It is proposed, [16, 17, 18], to change the viscosity $\boldsymbol{\nu}$ into a multiplier $\boldsymbol{\alpha}$ so defined that the evolution keeps constant "enstrophy" $\mathcal{D}(\mathbf{u}) \stackrel{def}{=} \sum_{\mathbf{k}} \mathbf{k}^2 \mathbf{u}^2_{\mathbf{k}}$ ("enstrophy thermostat"). Achieved by:

$$\boldsymbol{\alpha}(\mathbf{u}) = \frac{\sum_{\mathbf{c}} \sum_{\mathbf{k}} (-\mathbf{t}^{\mathbf{c}}_{\mathbf{k}}(\mathbf{u}) \mathbf{k}^2 \overline{\mathbf{u}}^{\mathbf{c}}_{\mathbf{k}} + \mathbf{k}^2 \mathbf{f}^{\mathbf{c}}_{\mathbf{k}} \overline{\mathbf{u}}^{\mathbf{c}}_{\mathbf{k}})}{\sum_{\mathbf{c}} \sum_{\mathbf{k}} \mathbf{k}^4 |\mathbf{u}^{\mathbf{c}}_{\mathbf{k}}|^2} \qquad (* : \mathbf{RNS}^{\mathbf{N}})$$

where $\mathbf{t}^c_{\mathbf{k}}(\mathbf{u})$ comes from the non-linear term in the eq.(#).

The **R** *in the name* RNS *stands to stress that the equation* $\mathbf{RNS}^{\mathbf{N}}$ *is time reversible, unlike the irreversible* $\mathbf{INS}^{\mathbf{N}}$*.*

Alternatively it is possible to fix α so that the energy $\mathcal{E}(\mathbf{u}) = \sum_c \sum_{\mathbf{k}} |u^c_{\mathbf{k}}|^2$ is exactly constant, which leads to a much simpler multiplier α:

$$\boldsymbol{\alpha}(\mathbf{u}) = \frac{\sum_c \sum_{\mathbf{k}} f^c_{\mathbf{k}} \overline{u}^c_{\mathbf{k}}}{\sum_c \sum_{\mathbf{k}} \mathbf{k}^2 |u^c_{\mathbf{k}}|^2} \qquad (** : \mathrm{ENS}^{\mathrm{N}})$$

studied in [19]. ("energy thermostat").

Physical interpr.: thermostats are forces with the effect of removing heat generated by the forcing.

For an incompressible fluid above, heat has to be taken away (in either enstrophy or in energy thermostat) to maintain the relation between pressure and temperature at constant density prescribed by the equation of state.

The stationary distributions for the equation, referred as RNS^N, with α in (*) are parameterized by the enstrophy value D as $\boldsymbol{\lambda}_{\mathbf{D}}^{\mathbf{N}}$ and their collection will be called "enstrophy ensemble", $\mathcal{E}^N_{enstrophy}$.

Likewise the stationary distributions for the equation, referred as ENS^N, with α in (**) are parameterized by the value E as $\boldsymbol{\vartheta}_{\mathbf{E}}^{\mathbf{N}}$ and form the "energy ensemble" $\mathcal{E}^N_{energy}$.

Given viscosity $\boldsymbol{\nu}$ suppose, for simplicity, that there is only one SRB distribution $\boldsymbol{\mu}^N_{\boldsymbol{\nu}} \in \mathcal{E}^N_{viscosity}$ for all N large:

Conjecture: *Let $D = \boldsymbol{\mu}^N_{\boldsymbol{\nu}}(\mathcal{D})$ be the average enstrophy. Then also the distribution $\boldsymbol{\lambda}^N_D \in \mathcal{E}^N_{enstrophy}$ is unique. The distributions $\boldsymbol{\mu}^N_{\boldsymbol{\nu}}, \boldsymbol{\lambda}^N_D$ are equivalent in the sense*

$$\lim_{N\to\infty} \boldsymbol{\mu}^N_{\boldsymbol{\nu}}(O) = \lim_{N\to\infty} \boldsymbol{\lambda}^N_D(O) \qquad (@)$$

for all local observables O.

In other words the viscosity ensemble and the enstrophy ensembles are equivalent in the limit $N \to \infty$ provided their entrophies agree, if the stationary distr. is unique.

More generally the conjecture is interpreted as saying that the SRB distributions for the INS^N equation can be put in one-to-one correspondence with the distributions for the RNS^N equation with the same enstrophy so that for corresponding distributions @ holds.

First test is a non trivial consequence: namely if both sides of INS^N or RNS^N are multipled by $\overline{u}^c_{\mathbf{k}}$ and summed over $c, \mathbf{k}$ one finds, respectively:

$$\frac{d}{dt}\mathcal{E}(\mathbf{u}) = -\nu\mathcal{D}(\mathbf{u}) + \mathbf{f}\cdot\mathbf{u}, \qquad \frac{d}{dt}\mathcal{E}(\mathbf{u}) = -\alpha(\mathbf{u})D + \mathbf{f}\cdot\mathbf{u}$$

(**no** non-linear term: energy conservation if $\mathbf{f} = \vec{0}, \boldsymbol{\nu} = 0$).

Equivalence condition is $\langle\, \mathcal{D}\,\rangle^N_{\boldsymbol{\nu}} = D$ and $O = \mathbf{f}\cdot\mathbf{u}$ is a local observable : hence it follows that the averages $\langle\, \mathbf{f}\cdot\mathbf{u}\,\rangle$ must be equal in the limit $N \to \infty$ and

$$\boldsymbol{\nu} = \lim_{N\to\infty} \langle\, \boldsymbol{\alpha}\,\rangle^N_D$$

because the averages of $\frac{d}{dt}\mathcal{E}(\mathbf{u})$ must vanish.

For $\mathbf{ENS^N}$ relevant tests are proposed in [19].

Comments

(1) Equivalence test: $\langle\, \alpha \,\rangle_D^N \xrightarrow[N\to\infty]{} \nu$ is performed in 2D and 3D: with positive results in all published cases: see Fig.4 in [16] and Fig.1 in [20], Fig.4 in [21], Fig.15a in [18].

(2) The 2D tests have shown that in many cases equivalence holds also for observables that are **non local**. Remarkable is the observable $\boldsymbol{\alpha}(\mathbf{u})$ studied as an observable for the INS^N equation. It also averages to $\boldsymbol{\nu}$ while presenting smaller fluctuations compared to the RNS^N, see [16] 2D case and 3D: Fig.16a [18] with exception in Fig.4 of [21];

(3) Remark (2) led to equivalence tests of other typically non local observables. A few tests, only in 2D so far, have been performed comparing, under the equivalence condition, the spectra of the symmetric part $J(\mathbf{u})$ **of the** $D_N \times D_N$ **Jacobian matrix** $\frac{\partial \dot{u}^c_{\mathbf{k}}}{\partial \dot{u}^b_{\mathbf{h}}} \stackrel{def}{=} J(\mathbf{u})_{c,\mathbf{k};b,\mathbf{h}}$**.**

Such observables are related to the Lyapunov exponents, [22, 23]. The result has been that essentially the eigenvalues *averaged over the flows* **agree if ordered in the same way (***e.g.* **in decreasing order):** see Fig.7 in [20] and Fig.5 in [16].

Most remarkable is that, while the average of the eigenvalues agree surprisingly well, the eigenvalues of the $J(\mathbf{u})$ reach equal averages, along the two evolutions, in spite of much larger fluctuations in the RNS^N compared to the INS^N, see Fig.6 in [20].

(4) The 3D tests are still somewhat preliminary: yet yield important informations. If the conjecture is correct it is expected that in RNS^N the fluctuating viscosity α fluctuates considerably and events in which $\alpha < 0$ occur.

Otherwise it can be proved that D being bounded ($\nu\langle\,\mathcal{D}\,\rangle^N \xrightarrow[N\to\infty]{} = \varepsilon < \infty$, at fixed f) it would follow that the velocity u remains smooth with all derivatives bounded uniformly in N, see [18, Appendix], **thus giving a new prespective to the question of existence and regularity of the NS flows**.

It is surprising, if ν is so small that the fluid is certainly in a turbulent regime, that for N large velocity fields $\mathbf{u}(t)$ with $\alpha(\mathbf{u}(t)) < 0$ are not observed (after a short transient time depending on the initial data) in several 3D simulations [21, 18].

Question to be understood is whether events with $\alpha < 0$ are not seen because they are rare events (which is my expectation), so rare to be missed (when N is large) in time series with too large time and/or integration step. For evidence, see Fig.15 in [18].

(5) Results in [18] suggest that conjecture above is too strong and might fail unless the definition of local observable is *deeply modified* **restricting the notion of local observable, for the purpose of the conjecture.**

So far the requirement for locality is that O **depends only on a finite number of harmonics** $\mathbf{u_k}$**: hence equivalence would be claimed for** O **depending on a single Fourier component k with** $|\mathbf{k}| > k_\nu$ *if* N *is large enough*.

BUT..

But from [18] it emerges that equivalence is not verified in several such tests: a further condition appears needed, *i.e.* **that** O **depends only on the components** $\mathbf{u_k}$ **with** $|\mathbf{k}| < c_0 k_\nu$ **for some constant** c_0 **of order** 1.

See "Conjecture 2" and Fig.11–13 in [18] which suggest a value $c_0 \sim \frac{1}{8}$. The evidence is not yet conclusive, in my view: more detailed analysis is needed to exclude $c_0 = \infty$.

(6) Tests of existence of several attractors have shown that even in presence of Chaotic motions there are cases in which multiple attractors can coexist showing strong intermittency phenomena., see figure below.

(7) The remark (4) suggests that the theory of the NS equation based on searching for existence and uniqueness in function spaces could be usefully extended to equivalent equations:

while not simplifying the problem it can open perspectives, just like introducing new equilibrium ensembles does not solve basic problems of SM but, actually, introduces new ones overcompensated by the deeper understanding of thermodynamics.

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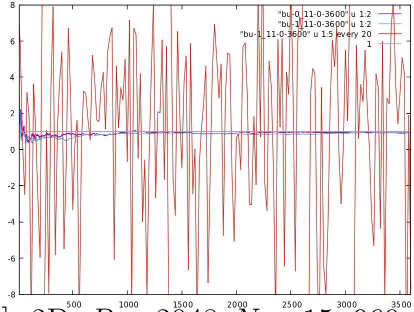
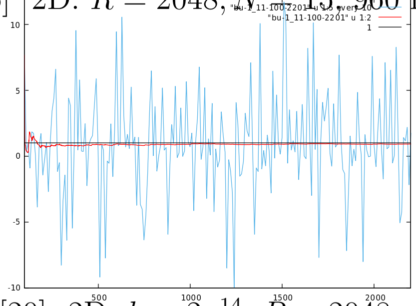


$\frac{\alpha(t)}{\nu}$, and $\frac{\langle \alpha \rangle_0^t}{\nu}$

Fig.4 [16] 2D: $R = 2048$, $N = 15,960$ modes, $h = 2^{-17}$



$\frac{\alpha(t)}{\nu}$ $\frac{\langle \alpha \rangle_0^t}{\nu}$

Fig.1 in [20] 2D: $h = 2^{-14}$, $R = 2048$, $N = 31, 3968$ modes

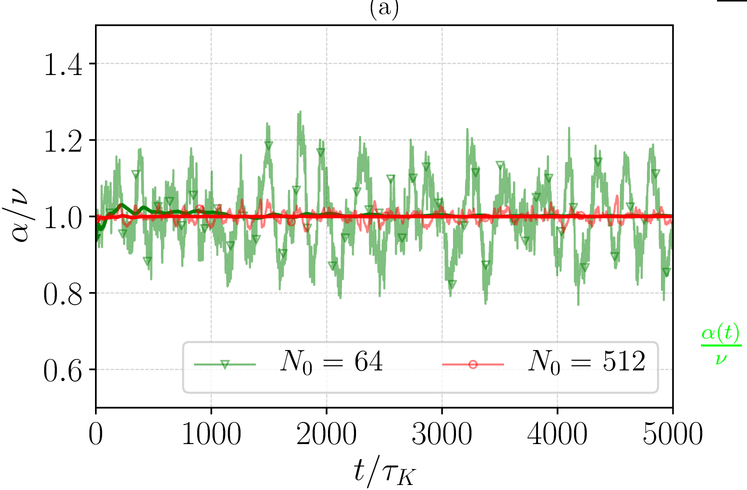


Fig.15a in [18] 3D $\frac{\alpha(t)}{\nu}$, and $\langle \frac{\alpha}{\nu} \rangle_0^t$ and ?? $\alpha > 0$??

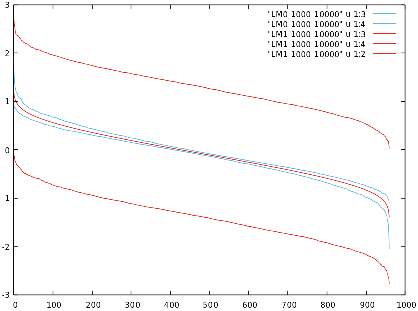


Fig.6 in [16] red=max,min $\overline{\lambda}_k$ in RNS
green=max,min in INS,
average in BOTH cases= central red

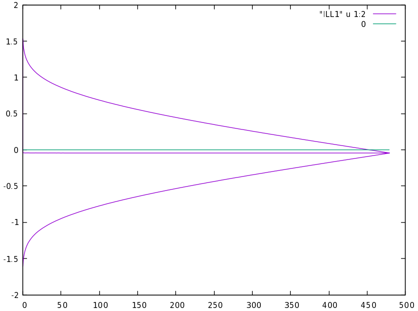
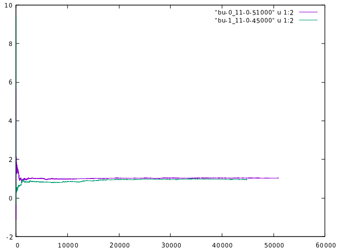
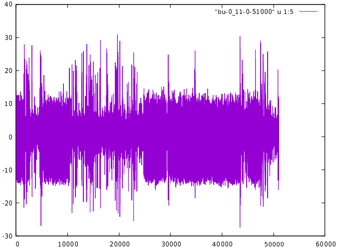


Fig.7 [20] local spectrum
RNS & INS **overlap**: $R = 2048$, $h = 2^{-13}$, $N = 15,960$
modes, $\sim$**Pairing** ?

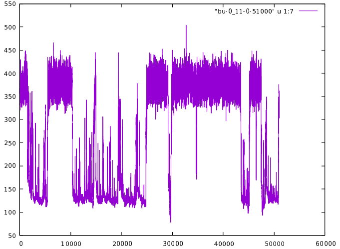


$\left\langle \frac{\alpha}{\nu} \right\rangle_0^t$

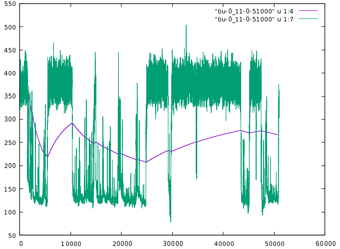
RNS & INS $R = 2048$, $h = 2^{-13}$, $N = 10$,



$\frac{\boldsymbol{\alpha}(t)}{\boldsymbol{\nu}}$



$\mathbf{En(t)}$



$\langle \mathbf{En} \rangle_0^t \quad \mathbf{En(t)}$

RNS & INS $R = 2048$, $h = 2^{-13}$, $N = 10$,