Stochastic Quantisation of Yang-Mills

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Cross fertilisation between Physics and Mathematics

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Euclidean QFT

Basic construction: Consider a functional S (action) on a space of fields. Euclidean QFT boils down to constructing the measure

$$\mu_\beta(D\varphi) = e^{-\beta S(\varphi)}\, D\varphi \;.$$

Above expression completely formal since Lebesgue measure $D\varphi$ on space of fields makes no sense. Hope that it yields a well-defined probability measure by some approximation procedure if S is coercive enough.

Interpretation as Gibbs measure for statistical mechanics model.

Example discussed today: Yang-Mills ($d = 2, 3$).

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Yang-Mills field theory

Setting (simplified): Fix compact Lie group G with Lie algebra $\mathfrak{g}$ (structure group). Fields: $\mathfrak{g}$-valued one-forms A on the torus $\mathbf{T}^d$ (actually G-equivariant connections on $\mathbf{T}^d \times G$).

Action S: L^2-norm of curvature tensor

$$S(A) = \int \|F^A(x)\|^2 \, dx \,, \qquad F^A_{ij}(x) = \big(\partial_i A_j - \partial_j A_i\big)(x) + [A_i, A_j](x) \,.$$

Distinguishing feature: action of gauge group $\mathcal{G} = \mathcal{C}^\infty(\mathbf{T}^d, G)$ onto A by

$$(g, A) \mapsto A^g = gAg^{-1} - (dg)g^{-1} \,,$$

such that F^A (and therefore $S(A)$) is invariant under this action.

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Gauge invariance

Problem: The action functional S is flat in the (infinitely many) directions in which $\mathcal{G}$ acts! There is no Lebesgue measure in infinite dimensions $\Rightarrow$ hints that there exists no measure on any space of equivariant connections that is invariant under the action of $\mathcal{G}$.

Good news: All physical observables $A \mapsto O(A)$ are gauge-invariant, namely $O(A^g) = O(A)$ for every $g \in \mathcal{G}$. Wilson loops: for loop $\gamma : [0,1] \to \mathbf{T}^d$ and class function $h \colon G \to \mathbf{R}$, define $O_{\gamma,h}(A) = h(\hat{\gamma}(1)\hat{\gamma}(0)^{-1})$ for $\hat{\gamma}$ the horizontal lift of γ to $\mathbf{T}^d \times G$.

Only need to build the measure μ on quotient space of gauge orbits.

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In $3D$: Approach inspired by Feldman, Glimm–Jaffe, etc. Series of works by Balaban, by Federbush, and by Magnen–Rivasseau–Sénéor ($4D$). No clear understanding of what the state space and observables are. Candidate state space by Cao–Chatterjee ('22).

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Stochastic quantisation

Proposed by Parisi & Wu '81, earliest rigorous works by Jona-Lasinio & Mitter '85.

Basic idea: Consider discrete approximation to Gibbs measure $e^{-\beta S(\varphi)}\, D\varphi$. This is invariant for stochastic evolution

$$d\varphi = -\nabla S(\varphi)\, dt + \sqrt{2/\beta}\, dW \;,$$

for W a Brownian motion with covariance structure adapted to the metric determining the gradient ∇.

Hope: Maybe one can pass to the limit for the dynamic?

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Quantisation equation

In case of Yang–Mills, this procedure yields

$$\partial_t A = -d_A^* F_A + \xi = -d_A^* d_A A + \frac{1}{2} d_A^* [A, A] + \xi \;,$$

Not parabolic! DeTurck–Donaldson trick: adding $d_A H(A)$ formally preserves dynamic on gauge orbits for any H. Choice $H(A) = -d_A^* A$ yields parabolic system. (Removes $-\partial_{ij}^2 A_j$ and changes l.o.t. in an inessential way)

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$$\partial_t A_i = \Delta A_i - \partial^2_{ij} A_j + \text{l.o.t.} + \xi_i \ .$$

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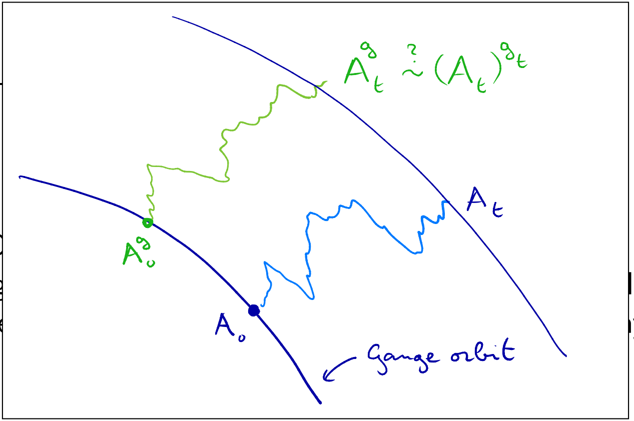

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Basic questions: … equivariance?

Problem

Equation of the form

$$\partial_t A = \Delta A + B(A, DA) + T(A, A, A) + \xi \ .$$

Solution to linear equation distribution-valued, so B and T meaningless a priori.

Natural approximation: replace ξ by ξ_ε, smooth at scale ε. Heuristic arguments suggest no convergence. Renormalisation needed, should be of the form

$$\partial_t A_\varepsilon = \Delta A_\varepsilon + B(A_\varepsilon, DA_\varepsilon) + T(A_\varepsilon, A_\varepsilon, A_\varepsilon) - C_\varepsilon A_\varepsilon + \xi_\varepsilon \ .$$

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Some results in $2D$

Theorem (Chandra, Chevyrev, H., Shen, '20): Can find Banach space Ω_α of distributional $\mathfrak{g}$-valued 1-forms and space $\mathcal{G}_\alpha$ of Hölder continuous gauge transformations such that:

1. For every fixed $C_\varepsilon = C$, one has $A_\varepsilon \to A$ in probability in $\mathcal{C}(\mathbf{R}_+, \Omega_\alpha)$ (modulo possible blow-up).
2. Smooth connections dense in Ω_α and quotient space $\mathcal{O}_\alpha = \Omega_\alpha / \mathcal{G}_\alpha$ is Polish.
3. Wilson loop observables continuous on Ω_α and $\mathcal{G}$-invariant.
4. Unique choice of C (but depending on smoothening of ξ!) such that the quotient process is Markov on $\mathcal{O}_\alpha$.
5. The process A on $\mathcal{O}_\alpha$ admits at most one invariant measure.

Conjecture: The process A has an invariant measure which coincides with the measure constructed by Sengupta, Lévy & al.

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Conjecture: The process A has an invariant measure which coincides with the measure constructed by Sengupta, Lévy & al.

Problems in $3D$

Some problems when trying to extend this to $3D$:

1. Explicit expression for C_ε intractable which is problematic for uniqueness argument. ($C_\varepsilon \sim c_1/\varepsilon + c_2 \log\varepsilon + c_3$ with c_2 intractable.)
2. Wilson loop observables not expected to exist. (Blow up already for Abelian case, cannot restrict free field to smooth line in $3D$.)
3. Limiting process A belongs to $\mathcal{C}^\beta$ for $\beta < -\frac{1}{2}$, but even solutions to deterministic Yang-Mills heat flow only exist for all i.c. in $\mathcal{C}^\beta$ when $\beta > -\frac{1}{2}$.
4. Natural gauge transformations associated to $\mathcal{C}^\beta$ are of regularity $\beta + 1$ (because of the term $(dg)g^{-1}$) but $\mathcal{G}_{\beta+1}$ acts continuously on $\mathcal{C}^\beta$ if and only if $\beta > -\frac{1}{2}$.

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Partial result in $3D$

Theorem (Chandra, Chevyrev, H., Shen, '21): Can find (non-linear) metric space $\mathcal{S}$ of distributional $\mathfrak{g}$-valued 1-forms such that

1. There exists a choice of C_ε such that $A_\varepsilon \to A$ in probability in $C(\mathbb{R}_+, \mathcal{S})$ (modulo possible blow-up).
2. Smooth connections dense in $\mathcal{S}$.
3. YM heat flow well-posed on $\mathcal{S}$ $\Rightarrow$ natural notion of gauge equivalence.
4. Unique choice of C_ε (modulo sequence converging to 0) such that A satisfies suitable notion of gauge equivariance.

Remark: Space $\mathcal{S}$ included in the state space of Cao–Chatterjee.

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- Link to lattice gauge theories?
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Happy Birthday Gianni!!

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