Proposta per un corso di dottorato (INDAM)

Lucia Di Vizio

May 2025

Classical Galois theory of algebraic equations has been extended to a Galois theory of linear differential equations [vS03] and more recently to the Galois theory of various kind of linear difference equations [vS97, DV21]. Apart from an intrinsic interest, the Galois theory of linear difference equations has been proven to have surprising applications in many different domains.

First, I will start giving an introduction to the Galois theory of linear differential equations. Then I will continue presenting the Galois theory of linear functional equations and some of its applications, that have been the object of recent publications. The interactions between combinatorics, probability and Galois theory of functional equations is very promising: Many problems are still open and for this reason I think that it is a good subject for a post-graduate class.

Here is a selection of applications that may be of interests for the audience of these lessons:

Differential transcendence of solutions of linear difference equations. In the spirit of Hölder's theorem, which says that the Gamma function is *differentially transcendental over the rational functions* (i.e., it is not solution of an algebraic differential equation with rational coefficients), there are several results on the fact that solutions of linear difference equations cannot be solutions of algebraic differential equations and viceversa. There have been several works in this directions in the last 30 years, culminating in the very general theorem [ADH21, Thm. 1.2]. The latter can be express in quite simple terms (in 3 of the 4 cases considered in *loc.cit.*) as follows:

Theorem 1 *Let $f \in \mathbb{C}((t))$ be a Laurent series satisfying a linear functional equation of the form*

$$\alpha_0 y + \alpha_1 \tau(y) + \cdots + \alpha_n \tau^n(y) = 0,$$

where $\alpha_i \in \mathbb{C}(t)$, not all zero, and τ is one of the following operators:

- $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$;
- $\tau(f(t)) = f(qt)$ *for some $q \in \mathbb{C}^*$, not a root of unity;*
- $\tau(f(t)) = f(t^m)$ *for some positive integer m.*

Then either $f \in \mathbb{C}(t^{1/r})$ for some positive integer r, or f is differentially transcendental over $\mathbb{C}(t)$. Moreover, in the case of the first operator, r is necessarily equal to 1.

To present the proof of this theorem, I'll need to introduce the so called linear differential groups *à la Kolchin*. Very roughly they are subgroups of GL_n defined by an ideal of differential polynomial. The definition is reminiscent of the theory of linear algebraic group, but quickly the theory develops in unexpected directions.

Klazar's theorem on Bell numbers, Yeliussizov-Pak conjecture and strong differential transcendence. In [Kla03], Klazar considers the ordinary generating function (OGF) of the Bell numbers $\phi(t) := 1 + \sum_{n\geq 1} \phi_n t^n$, where ϕ_n is the number of partitions of a set of cardinality $n \geq 1$, and proves that $\phi(t)$ is differentially transcendental over the field $\mathbb{C}(\{t\})$ of the germs of meromorphic functions at 0. To do so, he uses a functional equation satisfied by $\phi(t)$, namely:

$$\phi\left(\frac{t}{1+t}\right) = t\phi(t) + 1. \tag{1}$$

A classical and important property of the Bell numbers is that their *exponential generating function* (EGF)

$$\hat{\phi}(t) := 1 + \sum_{n\geq 1} \frac{\phi_n}{n!} t^n$$

satisfies

$$\hat{\phi}(t) = \exp(\exp t - 1). \tag{2}$$

Starting from the example of the Bell numbers, Pak and Yeliussizov formulated the following ambitious conjecture as an "advanced generalization of Klazar's theorem":

Conjecture 2 ([Pak19, Open Problem 2.4]) *If for a sequence of rational numbers $(a_n)_{n\geq 0}$ both ordinary and exponential generating functions $\sum_{n\geq 0} a_n t^n$ and $\sum_{n\geq 0} a_n \frac{t^n}{n!}$ are D-algebraic, then both are D-finite (equivalently, $(a_n)_{n\geq 0}$ satisfies a linear recurrence with polynomial coefficients in $\mathbb{Q}[n]$).*

In [BDVR24] A. Bostan, K. Raschel and myself have proved that all OGF whose EGF is defined by a "closed exponential form", similar to (2), satisfy a linear τ-equations for the operator $\tau(f(t)) = f\left(\frac{t}{1+t}\right)$. We produced a long list of known special EGF of this kind from the classical literature. In particular many of them satisfy an equation of order one with an inhomogeneous term, as in (1): We prove that they are differentially trasncendental over $\mathbb{C}(\{t\})$. This provides a long list of special series that confirms the conjecture above.

Applications to the study of the nature of Green functions on self-similar graphs. In collaboration with G. Fernandes and M. Mishna [DVFM23], we have proved the following results. We define the following field endomorphism of $\mathbb{C}((t))$:

$$\Phi_R : \sum_n f_n t^n \mapsto f(R(t)) := \sum_n f_n R(t)^n \,,$$

where R satisfies:

$$(\mathcal{R}) \qquad \begin{array}{l} R(t) \in \mathbb{C}(t),\ R(0) = 0,\ R'(0) \in \{0, 1, \text{roots of unity}\}, \\ \text{but no iteration of } R(t) \text{ is equal to the identity.} \end{array}$$

Our first main theorem is:

Theorem 3 *Let $R(t)$ satisfy assumption $(\mathcal{R})$. We suppose that there exist $a, b \in \mathbb{C}(t)$, and $f \in \mathbb{C}((t))$ such that $f(R(t)) = a(t)f(t) + b(t)$. Then either f is differentially transcendental over $\mathbb{C}(t)$ or there exists $\alpha, \beta \in \mathbb{C}(t)$ such that $f' = \alpha f + \beta$.*

For particular choices of a and b we obtain stronger results, namely:

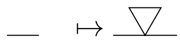
Theorem 4 *Let $R(t)$ satisfy assumption $(\mathcal{R})$. We suppose that there exists $b \in \mathbb{C}(t)$ and $f \in \mathbb{C}((t))$ such that $f(R(t)) = f(t) + b(t)$. Then either $f \in \mathbb{C}(t)$, or f is differentially transcendental over $\mathbb{C}(t)$.*

and:

Theorem 5 *Let $R(t)$ satisfy assumption $(\mathcal{R})$. We suppose that there exist $f \in \mathbb{C}((t))$ and $a \in \mathbb{C}(t)$, such that $f(R(t)) = a(t)f(t)$. Then either f is algebraic over $\mathbb{C}(t)$ and there exists a positive integer N such that $f^N \in \mathbb{C}(t)$, or f is differentially transcendental over $\mathbb{C}(t)$.*

As an example of applications, we present the result for Sierpiński graph, but similar conclusions are true for the entire class of *symmetric self-similar graphs*, as described by Böhn and Teufl [KT04].

The Sierpiński graph results from a fractal generating process starting with a single line, iteratively rewritten and rescaled in particular way. More precisely, one starts with a unit line, $S_0 =$ __ and applies the following replacement rule:


(3)

in an iterated process. Figure 1 demonstrates the first few iterates. The *Sierpiński graph* is the limit of this process.

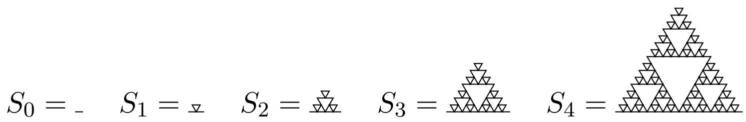


Figure 1: Initial iterates defining the Sierpiński graph.

The *Green function of a graph* is a probability generating function which describes the n-step displacement starting and returning to a certain origin vertex. The Green function of symmetric self-similar graphs satisfy homogeneous iterative equations,

$$G(R(t)) = a(t)G(t)$$

with algebraic (often rational) R and rational a. Roughly, the substitution $t \mapsto R(t)$ has a combinatorial interpretation reflecting the self-similarity of the graph [KT04]. As the graph is 4-regular, $G(4t)$ is the generating function for walks that begin and end at the origin on the Sierpiński graph. These walks are also known as *excursions* on the graph. The series begins:

$$G(4t) = 1 + 4\,t^2 + 4\,t^3 + 32\,t^4 + 76t^5 + 348\,t^6 + 1112\,t^7 + O(t^8).$$

Figure 2 illustrates an example excursion.

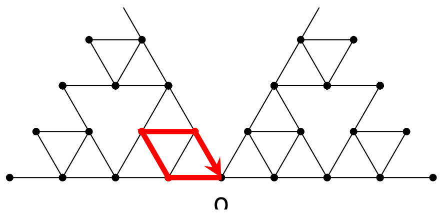


Figure 2: A close up on the origin (labelled o) of the Sierpinski Graph. The (red) path in bold is one of the 32 excursions of length 4.

Grabner and Woess [GW97, Proposition 1] proved that the Green function $G(t)$ for walks that return to their origin on the Sierpiński graph satisfies the functional equation

$$G\left(\frac{t^2}{4-3t}\right) = \frac{(2+t)(4-3t)}{(4+t)(2-t)}\,G(t). \tag{4}$$

We apply Theorem 5 to $G(t)$.

Theorem 6 *The Green function $G(t)$ of walks that start and end at the origin on the infinite Sierpiński graph is differentially transcendental over $\mathbb{C}(t)$.*

In light of Equation (4), to prove Theorem 6, it suffices to show that $G(t)$ is not algebraic. Grabner and Woess in *loc.cit.* show that the coefficient of t^n in $G(t)$ grows asymptotically like

$$n^{-\log 3/\log 5} F(\log n/\log 5)$$

as n goes to infinity, for some nonconstant periodic function F. The constant $-\log 3/\log 5$ is related to the fractal dimension of the underlying structure. Since the exponent of n is not rational, $G(t)$ is not algebraic (see [FS09, Theorem VII.8]), hence it is differentially transcendental.

Similar functional equations appears in enumerative problems for complete trees and pattern avoiding permutations. Those results have been further generalized in [KM24].

References

[ADH21] Boris Adamczewski, Thomas Dreyfus, and Charlotte Hardouin, *Hypertranscendence and linear difference equations*, Journal of the American Mathematical Society **34** (2021), no. 2, 475–503. MR 4280865

[BDVR24] Alin Bostan, Lucia Di Vizio, and Kilian Raschel, *Differential transcendence of Bell numbers and relatives: A Galois theoretic approach*, American Journal of Mathematics **146** (2024), no. 6, 1577–1615.

[DV21] Lucia Di Vizio, *Difference Galois Theory for the "Applied" Mathematician*, Arithmetic and Geometry over Local Fields, Lecture Notes in Mathematics, vol. 2275, Springer Cham, 2021, pp. 29–59.

[DVFM23] Lucia Di Di Vizio, Gwladys Fernandes, and Marni Mishna, *Inhomogeneous order 1 iterative functional equations with applications to combinatorics*, November 2023.

[FS09] Philippe Flajolet and Robert Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[GW97] Peter J. Grabner and Wolfgang Woess, *Functional iterations and periodic oscillations for simple random walk on the Sierpiński graph*, Stochastic Processes and their Applications **69** (1997), no. 1, 127–138. MR 1464178

[Kla03] Martin Klazar, *Bell numbers, their relatives, and algebraic differential equations*, Journal of Combinatorial Theory, Series A **102** (2003), no. 1, 63–87.

[KM24] Yakob Kahane and Marni Mishna, *Differential transcendence and walks on self-similar graphs*, November 2024.

[KT04] Bernhard Krön and Elmar Teufl, *Asymptotics of the transition probabilities of the simple random walk on self-similar graphs*, Transactions of the American Mathematical Society **356** (2004), no. 1, 393–414. MR 2020038

[Pak19] Igor Pak, *Complexity Problems in Enumerative Combinatorics*, Proceedings of the International Congress of Mathematicians (ICM 2018) (Rio de Janeiro, Brazil), World Scientific, May 2019, pp. 3153–3180.

[vS97] Marius van der Put and Michael F. Singer, *Galois theory of difference equations*, Lecture Notes in Mathematics, vol. 1666, Springer-Verlag, Berlin, 1997.

[vS03] ———, *Galois theory of linear differential equations*, Springer-Verlag, Berlin, 2003.