Course Proposal

Title: Mean-Field Limits in Classical and Quantum Mechanics

Proposed period: May 2018

Syllabus:

The purpose of this course is to review recent mathematical results and methods used in the derivation of mean-field PDEs from the classical or quantum dynamics of large particle systems.

Describing the evolution of large particle systems with the motion equations based on the fundamental principles of physics is in general impossible in practice. For instance, writing Newton's second law for each gas molecule in 1 gram of argon would result in a system of about $2 \cdot 10^{23}$ coupled differential equations. Even if solving the Cauchy problem for such a large system of differential equations was possible, one would need to know precisely its initial data — i.e. the position and momentum of each gas molecule in the gas, which is of course impossible.

For that reason, one seeks to replace the first principle description of large particle systems (which one can think of as an axiom of physics) by reduced models governing the evolution of the "typical" particle in the system. In particular, the reduced model will involve few degrees of freedom — exactly as many as are involved in the description of a single particle.

The mean-field limit is one of the regimes where such reduced models can be derived rigorously from the fundamental principles of physics. The key idea in all mean-field models is that the typical single particle is driven by the potential created by the cloud of all the other particles, and that this potential can be expressed in terms of the single particle number density. In other words, the single particle density is driven by the self-consistent potential which it creates.

The mathematical justification of the mean-field limit involves very different ideas and methods. This course is aimed at giving a concise, and yet precise description of the main approaches to the problem of the mean-field limit in the field of mathematical analysis.

The main approaches used on this type of problem are:

(a) the method based on the formalism of the BBGKY hierarchy,
(b) the method based on the empirical measure,
(c) the method based on a direct analysis of the N-body Liouville equation,
(d) other methods.

One of the main purpose of the present work is to outline the respective merits of each of these methods on a few typical examples.

There is a considerable amount of literature on the subject of mean field limits. The course is focused on the rigorous derivation of mean field equations from the N-body problem in the context of classical mechanics, and in quantum mechanics.

The following topics will be discussed in detail in the course.

(1) Derivation of the Vlasov equation from the N-body problem in classical mechanics for Lipschitz continuous force fields (following Neunzert-Wick [10], Braun-Hepp [1], Dobrushin [2]);

(2) Same problem as (1) on the case of singular force fields (following the work of Hauray-Jabin [6] or Lazarovici-Pickl [8]);

(3) Derivation of the Hartree equation from the N-body problem in quantum mechanics (following Spohn [11]) for bounded potentials;

(4) Same problem as (3) in the case of singular potentials (following Erdös-Yau [3] or Pickl [9]);

(5) Uniformity in the Planck constant of the mean-field limit in quantum mechanics (following [4], [5].

Prerequisites:

The course does not assume any advanced knowledge in quantum or statistical mechanics. The physical meaning of the mathematical objects involved in the analysis of the mean-field limit will be discussed during the course, whenever necessary.

Some familiarity with the theory of evolution PDEs will be assumed, together with some basic knowledge on functional analysis and operator theory (especially the definition of compact, Hilbert-Schmidt and trace-class operators).

References

[1] W. Braun, K. Hepp: *The Vlasov dynamics and its fluctuations in the1/N limit of interacting classical particles*; Commun. Math. Phys. 56, (1977), 101–113 .

[2] R. Dobrushin: *Vlasov equations*, Funct. Anal. Appl. **13** (1979), 115–123.

[3] L. Erdös, H.-T. Yau: *Derivation of the nonlinear Schrdinger equation from a many body Coulomb system.* Adv. Theor. Math. Phys. **5** (2001), 1169–1205.

[4] F. Golse, C. Mouhot, T. Paul: *On the Mean-Field and Classical Limits of Quantum Mechanics*, Commun. Math. Phys. **343** (2016), 165–205.

[5] F. Golse, T. Paul: *The Schrödinger Equation in the Mean-Field and Semiclassical Regime*, Arch. Rational Mech. Anal. **223** (2017), 57–94.

[6] M. Hauray, P.-E. Jabin: *Particle approximations of Vlasov equations with singular forces*, Ann. Sci. Ecol. Norm. Sup. **48** (2015), 891–940 .

[7] D. Lazarovici: *The Vlasov-Poisson dynamics as the mean-field limit of rigid charges.* (preprint). arXiv:1502.07047

[8] D. Lazarovici, P. Pickl: *A mean-field limit for the Vlasov?Poisson system.* (preprint).arXiv:1502.04608

[9] P. Pickl: *A simple derivation of mean field limits for quantum systems.* Lett. Math. Phys. **97** (2011), 151–164.

[10] H. Neunzert, J. Wick: *Die Approximation der Lsung von Integro-Differentialgleichungen durch endliche Punktmengen.* Lecture Notes in Mathematics, vol. 395, pp. 275?290. Springer, Berlin (1974)

[11] H. Spohn: *Kinetic equations from hamiltonian dynamics.* Rev. Mod. Phys. **52** (1980), 600–640.