DIFFERENTIAL EQUATIONS AND QUANTUM GROUPS

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Quantum groups are deformations of the groups of matrices one frequently encounters in Linear Algebra and Differential Geometry, such as invertible $n \times n$ matrices and orthogonal ones. They were discovered in the mid–eighties in the study of 1 and 2–dimensional models in Statistical Mechanics describing, for example, thin layers of ice. They have since had a profound impact on many areas of Mathematics and Mathematical Physics: Representation Theory, Knot Theory and Low–Dimensional Topology, Algebraic and Enumerative Geometry, String Theory and Conformal Field Theory.

The aim of this course will be to explain how quantum groups can be used to solve a class of problems which has occupied mathematicians since the 19th century, namely the computation of the monodromy of differential equations in the complex domain. It will focus on two main examples: the Knizhnik–Zamolodchikov and Casimir equations. The monodromy of these equations gives rise to representations of Artin's group of braids on n strings and its generalisations. In addition to their intrinsic interest, these representations encode a variety of important information: knot invariants in Low–Dimensional Topology, multivaluedness of conformal blocks in Conformal Field Theory, and Stability Conditions in Algebraic Geometry.

The course will cover a blend of Representation Theory (quantum groups), Category Theory (braided tensor categories), and Deformation Theory (Etingof–Kazhdan quantisation, and Hochschild cohomology). It should be of interest to students with a background, or interest in Algebra, Geometry or Mathematical Physics.

The course will start at the end of September and run until the mid November. The instructor will be available on campus two weeks before the start of classes for students wishing to have more information and/or do some background reading.

Prerequisites. The presentation will be adapted to the mathematical sophistication of the audience. It will only assume some familiarity with the subjects listed below. Each is accompanied by one or two references to direct interested students.

(1) Basic Differential Geometry (the language of Connections, Curvature and Holonomy), [12, chaps. I–II].
(2) Basic Category Theory (the language of functors, and their natural transformations), [10, chap. 1] (optionally, chaps. VII and XI).
(3) Basic theory of Lie algebras (definition, enveloping algebras, PBW theorem).
(4) Complex semisimple Lie algebras (structure theory, root systems, Serre presentation), [13, chaps. I-VI] or [9, chaps. 1–19].

1. Detailed syllabus

1.1. **Differential equations on hyperplane complements.**
Holonomy equations of a hyperplane complement.
Main examples:

(1) The Knizhnik–Zamolodchikov (KZ) connection of a metric Lie algebra.
(2) The Casimir connection of a reductive Lie algebra.

1.2. **Braid groups.**
Artin's presentation of B_n.
Brieskorn's presentations of the generalised braid group B_W.

1.3. **The quantum group $U_\hbar\mathfrak{g}$.**
The Drinfeld–Jimbo quantum group $U_\hbar\mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$.
Representations of $U_\hbar\mathfrak{g}$.
$U_\hbar\mathfrak{g}$ as a bialgebra.
Universal R–matrix. Braid group representations.
Statement of the Drinfeld-Kohno theorem.
q–triple exponentials and Lusztig's quantum Weyl group.
Statement of the monodromy theorem for the Casimir equations.

1.4. **Braided tensor categories and quasi–triangular quasi–hopf algebras.**
Braided tensor categories and quasitriangular quasibialgebras.
The monodromy of the KZ equations as commutativity and associativity constraints.
The Drinfeld–Kohno theorem as an equivalence of braided tensor categories.

1.5. **Lie bialgebras.**
Lie bialgebras as semiclassical limits of quantised universal enveloping algebras.
Quasi–triangular Lie bialgebras.
Manin triples and Drinfeld doubles.
Example: semisimple Lie algebras.
Drinfeld–Yetter modules.

1.6. **Etingof–Kazhdan quantisation.**
First iteration: quantisation of finite–dimensional Lie bialgebras.
Second iteration: quantisation of arbitrary Lie bialgebras.
PROPs.
Third iteration: the PROPs LBA and HA and universal quantisation of Lie bialgebras.
Fourth iteration: the PROPs DY_n and quantisation of representations of Lie bialgebras.
Cohomology of Schur functors and of $\{DY_n\}$.
Drinfeld–Kohno theorem via Etingof–Kazhdan quantisation.

1.7. **Quasi–Coxeter algebras and categories.**
Quasi–Coxeter algebras.
Quasi–Coxeter categories.
De Concini–Procesi wonderful models.
The monodromy of the Casimir connection as a quasi–Coxeter structure.
The monodromy theorem as an equivalence of quasi–Coxeter categories.

1.8. **Braided quasi–Coxeter categories.**
Quasi–Coxeter, quasitriangular quasibialgebras.
The quantum group $U_\hbar\mathfrak{g}$ as a qCqtqba.

1.9. **The differential twist and fusion operator.**
Braided quasi–Coxeter categories from differential twists.
The joint dynamical KZ and Casimir equations.
The fusion operator of a semisimple Lie algebra.

1.10. **Relative Etingof–Kazhdan functor.**
Split pairs of Lie bialgebras and Hopf algebras and the associated PROPs.
Relative EK quantisation.
Transfer of braided quasi–Coxeter structure from $U_\hbar\mathfrak{g}$ to $U\mathfrak{g}$.
Rigidity of braided quasi–Coxeter structures on $U\mathfrak{g}$.

References

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