An Introduction to the Theory of Viscosity Solutions

Graziano Crasta

The purpose of this course is to present the basic theory of viscosity solutions of fully nonlinear scalar partial differential equations of first and second order. Some applications of the basic theory to recent problems involving the infinite Laplace operator will be given in the final part of the course.

The plan is the following.

1. Motivations and examples
2. First order equations (Hamilton–Jacobi–Bellman equations). Generalized gradient. Sub- and super-solutions. Stability and comparison results for the Dirichlet problem.
3. Second order fully nonlinear equations. Semijets. Sub- and super-solutions. Comparison principles. Existence of solutions to the Dirichlet problems via Perron's method.
4. Some topics on the infinite Laplace equation.

References

[1] M. Bardi and I. Capuzzo Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Systems & Control: Foundations & Applications, Birkhäuser, Boston, 1997.

[2] P. Cannarsa and C. Sinestrari, Semiconcave functions, Hamilton-Jacobi equations and optimal control, Progress in Nonlinear Differential Equations and their Applications, vol. 58, Birkhäuser, Boston, 2004.

[3] M.G. Crandall, H. Ishii, and P.L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[4] N. Katzourakis, An Introduction To Viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in L^∞, Springer, 2015.

[5] S. Koike, A Beginner's Guide to the Theory of Viscosity Solutions, MSJ Memoir, vol. 13, Math. Soc. Japan, 2012.