The hard Lefschetz theorem beyond positivity, and applications to combinatorics

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Abstract.
Consider a simplicial complex that allows for an embedding into $\mathbb{R}^d$. How many faces of dimension $d/2$ or higher can it have? How dense can they be?

This basic question goes back to Descartes' "Lost Theorem" and Euler's work on polyhedra. Using it and other fundamental combinatorial problems, we introduce a version of the Kähler package beyond positivity, allowing us to prove the hard Lefschetz theorem for toric varieties (and beyond) even when the ample cone is empty. A particular focus lies on replacing the Hodge-Riemann relations by a non-degeneracy relation at torus-invariant subspaces, allowing us to state and prove a generalization of theorems of Hall and Laman in the setting of toric varieties and, more generally, the face rings of Hochster, Reisner and Stanley. We motivate this adventure into "beyond Kähler" algebraic geometry.

This has several applications to quantitative and combinatorial topology, among them the following:

- We fully characterize the possible face numbers of simplicial rational homology spheres, resolving the g-conjecture of McMullen in full generality and generalizing Stanley's earlier proof for simplicial polytopes.
- We prove that for a simplicial complex that embeds into $\mathbb{R}^{2d}$, the number of d-dimensional simplices exceeds the number of $(d-1)$-dimensional simplices by a factor of at most $d+2$. This generalizes a result going back to Descartes and Euler, and resolves the Grünbaum-Kalai-Sarkaria conjecture.

No prerequisites beyond a basic algebra course and basic algebraic topology.