A semi-Lagrangian scheme for the Game p-Laplacian

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Outline

p-Laplacians and applications

p-Laplacian and ∞-Laplacian operators enter in many mathematical problems and models for applications:

- Classical problem of Calculus of Variations such as the continuous extension of a given function
- Engineering mechanics: minimization of maximum stress or deflection
- Image processing (edge detection) and interpolation (inpainting)
- Granular materials: models for growing sandpiles can be derived as limits of fast/slow diffusion problems in terms of p-Laplacians
- Stochastic game theory: random-turn games can be used in economical and political modeling, real world conflicts where opposing agents continually seek to improve their positions through incremental *tugs*, the move sets are player-symmetric but independent of what the others do.

Variational p-Laplacians and p-harmonic functions

Definition

$$\Delta_p u := \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad (1 \le p < \infty)$$

$$\Delta_\infty u := \sum_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

They are *degenerate elliptic* for $2 < p \le \infty$, and *singular* for $1 \le p < 2$.

A *$p-$ harmonic function* in a bounded domain Ω will be a weak solution of the equation

$$\Delta_p u = 0 \ , \qquad (1)$$

that is a continuous function in the Sobolev space $W^{1,p}_{loc}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \phi)dx = 0, \qquad \forall \phi \in C_0^\infty(\Omega)$$

Given $p > 1$, a bounded domain Ω and a continuous $F : \partial\Omega \to \mathbb{R}$, solving the $p-$ *Dirichlet problem*

$$-\Delta_p u = 0 \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega, \tag{2}$$

means to find a continuous extension $u : \Omega \to \mathbb{R}$ of F which is p-harmonic, that is a function u that minimizes the energy functional $\int_\Omega |\nabla u|^p dx$ subject to the given boundary condition.

For $p = \infty$, given a Lipschitz function $F : \partial\Omega \to \mathbb{R}$, the solution of

$$-\Delta_\infty u = 0 \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega, \tag{3}$$

is a Lipschitz extension of F to Ω. It can be proven that it is also a minimal extension of F ($Lip_\Omega u = Lip_{\partial\Omega} u$) and an absolutely minimal extension ($Lip_U u = Lip_{\partial U} u \ \ \forall U \subseteq \Omega$)

As $p \to \infty$ the p-harmonic extensions of F converge to the solution of (3)

- Due to the degeneracy of the problem, the notion of *viscosity solution* is needed.
- For the variational p-Laplacian ($1 < p < \infty$), viscosity solution and weak solution are equivalent [Juutinen et al. '01].
- If $F \in W^{1,p}(\Omega)$, there exists a unique p-harmonic function u in Ω such that $u - F \in W_0^{1,p}(\Omega)$, and u minimizes $\int_\Omega |\nabla u|^p$ among all $v \in W^{1,p}(\Omega)$ such that $v - F \in W_0^{1,p}(\Omega)$ [Heinonen et al. '93].
- There exists a unique viscosity solution for problem (3) [Jensen '93, Barles-Busca '01, Aronsson-Crandall-Juutinen '03] .
- The following *regularity result* holds [Di Benedetto, '83]:
  If u is p-harmonic in Ω, then it is everywhere differentiable in Ω and real analytic whenever $\nabla u \neq 0$. Moreover, u has a Holder continuous gradient, that is $u \in C^{1,\gamma}(\Omega)$ for some $\gamma > 0$.
- In general, solutions are not twice differentiable:
  $u(x, y) = |x|^{4/3} - |y|^{4/3}$ is an example of an absolute minimizer in the square $\Omega = (-1, 1)^2$ (it solves (3) and is not C^2). [Aronsson, '67]

Game interpretation

- In the classical case ($p = 2$) the Dirichlet problem (2) can be solved by starting a Brownian motion B at x, running it until the hitting time τ of the boundary, and taking $u(x) = E_x[F(B(\tau))]$.
  The brownian motion can be derived as the continuum limit of random walks in Ω when the step lenght goes to zero.
- A similar interpretation can be given to the associated non-homogeneous problem ($-\Delta u = f$ in Ω) if we think to f as a sort of running cost for the random walk.
- Analogous interpretations are possible for the p-harmonic extensions in terms of the continuum limit of the values of certain games:
  - $p = 1$: minimum exit time problem
  - $p = \infty$: tug-of-war game
  - $1 < p < \infty$: tug-of-war with noise

Motion by mean curvature ($p = 1$) [Kohn-Serfaty, '06]

The equation for motion of level sets by mean curvature can be interpreted in terms of the following deterministic two player game. Let $\Omega \in \mathbb{R}^2$, $x \in \Omega$.

- Player 1 wants to reach the boundary, player 2 tries to obstruct him.
- At each step player 1 choses a direction $v \in \mathbb{R}^2$, player 2 replaces v with $\pm v$ (i.e. stand or reverse v), then player 1 moves from x to $x + \sqrt{2}\varepsilon bv$.
- The value function of the game is the minimum exit time, given by $u_\varepsilon(x) = \varepsilon^2 k$ if player 1 needs k steps to exit Ω starting from x and following an optimal strategy.
- For $\varepsilon \to 0$ (the continuum limit) u_ε converges to the solution of

$$-\Delta_1 u = 1 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Random turn games

Two-player zero-sum games

- X set of states, $Y \subset X$ nonempty set of terminal states (target)
- $F : Y \to \mathbb{R}$ terminal payoff function,
  $f : X \setminus Y \to \mathbb{R}$ running payoff function
- x_0 initial state
- E_1, E_2 transition graphs with vertex set X

The game:

- A token is initially placed at x_0
- At the k-th step a fair coin is tossed and the player who wins may move the token to any x_k s.t. (x_{k-1}, x_k) is a directed edge in the transition graph
- The game ends the first time $x_k \in Y$, with player 1's payoff

$$F(x_k) + \sum_{i=0}^{k-1} f(x_i)$$

- Player 1 seeks to maximize this payoff, Player 2 to minimize it

The tug-of-war game ($p = \infty$) [Peres-Schramm-Sheffield-Wilson '08]

In the conventional *tug-of-war* game:

- $E_1 = E_2 = E$, with E undirected (all moves are reversible)
- $Y = Y^1 \cup Y^2$, $F \equiv 1$ on Y^1, $F \equiv 0$ on Y^2
- no running payoff
- each player tries to "tug" the token to his own target and away from his opponent one
- the game ends when a target is reached
- the value of the game when the game starts at x is given by
  $u_1(x) = \sup_{S_1} \inf_{S_2} \hat{F}(S_1, S_2)$ for player 1,
  $u_2(x) = \inf_{S_2} \sup_{S_1} \hat{F}(S_1, S_2)$ for player 2,
  where S_1, S_2 are the strategies for the two players, and $\hat{F}$ denotes the expected total payoff at the termination of the game
- the game has a value u when $u_1(x) = u_2(x) = u(x)$

The tug-of-war game on a metric spaces

- (X, d) metric space, $Y \subset X$
- E_ε edge-set s.t. $x \sim y$ iff $d(x, y) < \varepsilon$
- $u^\varepsilon(x)$ value of the ε-t.o.w. game with terminal payoff F and running payoff $\varepsilon^2 f$ which starts at $x = x_0 \in X \setminus Y$:
  - at step k a coin is tossed and the winner choses x_k s.t. $d(x_k, x_{k-1}) < \varepsilon$
  - game ends when $x_k \in Y$, with payoff $F(x_k) + \varepsilon^2 \sum_{i=0}^{k-1} f(x_i)$
- $u(x) = \lim_{\varepsilon \to 0} u^\varepsilon(x)$ (if $\exists$) is the *continuum value* of the t.o.w. game

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $F : \partial\Omega \to \mathbb{R}$ uniformly continuous, $f : \Omega \to \mathbb{R}$ either zero or strictly positive uniformly continuous.
Then there exists a unique $u : \overline{\Omega} \to \mathbb{R}$ continuous viscosity solution of

$$-\Delta_\infty^G u = 2f \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega \tag{4}$$

which is the continuum value of the tug-of-war on (Ω, d, F, f). Here:

$$\Delta_\infty^G u := |\nabla u|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

The tug-of-war with noise $(1 < p < \infty)$ [Peres-Sheffield '08]

- $1 < p < \infty$, $x_0 \in \Omega$ starting position
- at each step k the player who wins the toss chooses a vector v_k s.t. $|v_k| \leq \varepsilon$ and set $x_k = x_{k-1} + v_k + z_k$, where z_k is a random noise vector orthogonal to v_k with lenght $\sqrt{1/(p-1)}|v_k|$ ($p \to \infty$: $|z_k| \to 0$, tug-of-war; $p = 2$: $|z_k| = |v_k|$, random walk)
- the game ends when the boundary is reached at some point y and player 1's payoff is set to $F(y)$ (Player 1 tries to reach the boundary, player 2 tries to prevent that)
- $u^\varepsilon(x)$ is the value of the game for player 1 starting from point $x \in \Omega$

The Tug-of-war game with noise

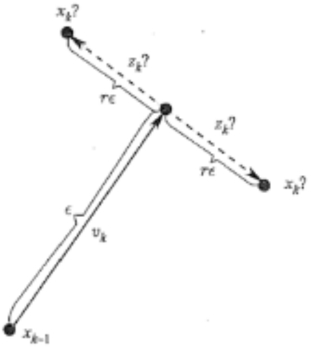

Figure 1. A move of tug-of-war with noise in dimension 2 for the noise distribution μ given by $\mu\{(0,r)\} = \mu\{(0,-r)\} = 1/2$. The player who wins the coin toss adds a vector v_k of length at most ϵ to the game position x_{k-1}, and then a random noise vector z_k with law μ_{v_k} (and magnitude $r|v_k|$) is added to produce x_k. In the figure, $|v_k| = \epsilon$.

The Tug-of-war game with noise

- If Ω is *game-regular* (*), as $\varepsilon \to 0$ the functions u^ε converge uniformly to the unique p-harmonic extension u of F

  (*) For any $y \in \partial\Omega$, if the game starts near y, player 1 has a strategy for making the game terminate near y with high probability.
  Sufficient conditions:
  - $p > d$ (in $\mathbb{R}^d$)
  - $\partial\Omega$ satisfies the cone property
  - Ω simply connected (in $\mathbb{R}^2$)

- If we add a running payoff of size $\varepsilon^2 f(x_k)$ at the k-th step, u will be the solution of problem

$$-\Delta_p^G u = 2f \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega \tag{5}$$

where $\Delta_p^G u := \dfrac{1}{p} |\nabla u|^{2-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$.

Game p-Laplacians

For $1 \le p < \infty$, it is defined by

$$\Delta_p^G u := \frac{1}{p} |\nabla u|^{2-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = \frac{1}{p} |\nabla u|^{2-p} \Delta_p u,$$

which is now singular for every $p \neq 2$ (and $\Delta_2^G u = \frac{1}{2}\Delta_2 u$).
By expanding the derivatives:

$$\Delta_p^G u = \frac{1}{p} \Delta_2 u + \frac{p-2}{p} |\nabla u|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{6}$$

then, by taking the limit for $p \to \infty$, we recover the previous definition of the *game* $\infty-$*Laplacian*:

$$\Delta_\infty^G u := |\nabla u|^{-2} \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = |\nabla u|^{-2} \Delta_\infty u.$$

Remarks

- When u is twice differentiable and $\nabla u \neq 0$,

$$\Delta_\infty^G u =< D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} >,$$

where $D^2 u$ denotes the Hessian matrix, that is the second derivative of u in the gradient direction.

- It can be easily seen that

$$\Delta_1^G u = |\nabla u| \Delta_1 u = \Delta_2 u - \Delta_\infty^G u,$$

that is the game 1-Laplacian can be thought as the second derivative of u in the orthogonal direction to ∇u.

- Combining previous relation with (6), we get the interesting characterization:

$$\Delta_p^G = \frac{1}{p} \Delta_1^G + \frac{1}{q} \Delta_\infty^G, \tag{7}$$

(q conjugate exponent of p), that is any game p-Laplacian can be thought as the convex combination of the two limiting cases.

- Since Δ_p^G and Δ_p differ only by the factor $p|\nabla u|^{p-2}$, the equation $\Delta_p^G u = 0$ is equivalent in the classical, weak and viscosity sense to the Euler-Lagrange equation (1) (if $\nabla u \neq 0$). Then when the homogeneous Dirichlet problem is treated, the distinction between the two operators is irrelevant.

- The viscosity solution definition has to be extended for the game p-Laplacian case, due to the singularity for any $p \neq 2$.

- We want to study the general Dirichlet problem

$$-\Delta_p^G u = f \quad \text{in } \Omega, \qquad u = F \quad \text{on } \partial\Omega \tag{8}$$

which has no variational sense, but a natural game-theoretic interpretation.

Viscosity solutions in $\mathbb{R}^2$

Definition

Given $1 < p \leq \infty$, an upper semi-continuous function [respectively, lower semi-continuous] $u : \Omega \to \mathbb{R}$ is a *viscosity subsolution [supersolution]* of

$$-\Delta_p^G u(x) = f(x) \text{ in } \Omega, \tag{9}$$

if for any $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum [local minimum] at $x \in \Omega$, we have

(i) $-\Delta_p^G \phi(x) \leq f(x) \quad [-\Delta_p^G \phi(x) \geq f(x)] \quad \text{if } \nabla\phi(x) \neq 0$;

(ii) $-\Delta_2^G \phi(x) \leq f(x) \quad [-\Delta_2^G \phi(x) \geq f(x)] \quad \text{whenever } \nabla\phi(x) = 0.$

Definition

A function u is a *viscosity solution* of (9) if u is a viscosity subsolution and supersolution according to (i) and (ii) .

Numerical schemes for the p-Laplacian problems

For the variational p-Laplacian several approximation schemes have been proposed:

- Finite element methods: Barrett-Liu '94
- Finite difference methods for degenerate second order operators: Crandall-Lions '96, Oberman '04
- Finite volumes methods: Andreyanov-Boyer-Hubert '06

Here we present a FD approach which is strictly connected with the game theory interpretation of the problems, and which then applies to both homogeneous (where variational and game Laplacian problems have the same solutions) and non-homogeneous cases.

The key tool is the notion of p-average.

Definition

Given a finite set of real numbers, $S = \{s_1, s_2, ..., s_m\}$, we denote by $A_p(S)$ the *p-average* of its elements, that is $A_p(S)$ is such that

$$\sum_{j=1}^{m} |s_j - A_p(S)|^p = \min_{c \in \mathbb{R}} \sum_{j=1}^{m} |s_j - c|^p \quad \text{if} \quad 1 < p < \infty, \qquad (10)$$

$$A_\infty(S) = \frac{1}{2} \left[\max_{s_j \in S} s_j + \min_{s_j \in S} s_j \right],$$

$$A_1(S) = median\ (S).$$

- By convexity $A_p(S)$ is uniquely defined for $1 < p \leq \infty$.
- If $s_1 \leq s_2 \leq .. \leq s_m$: $A_1(S) = \begin{cases} s_{k+1} & \text{if } m = 2k+1, \\ (s_k + s_{k+1})/2 & \text{if } m = 2k. \end{cases}$
- An easy calculation shows that $A_2(S)$ is the usual arithmetic mean

Properties of the $p - average$

- For any $k \in \mathbb{R}$: $A_p(S + k) = A_p(S) + k$
- $\min_{j=1..m} s_j \leq A_p(S) \leq \max_{j=1..m} s_j$
- Let $S = \{s_1, s_2, ..., s_m\}$ and $T = \{t_1, t_2, ..., t_m\}$ be two finite sets of real numbers having the same number m of elements, and let $1 \leq p \leq \infty$ be fixed. If $t_j \leq s_j$, $\forall j = 1..m$, then $A_p(T) \leq A_p(S)$.
- Let S and T be two finite sets of real numbers having the same number of elements, and let $1 \leq p \leq \infty$ be fixed. Assume that $S = \{s_1, s_2, ..., s_m\}$ and $T = \{t_1, t_2, ..., t_m\}$ verify $t_j = s_j + \delta_j$, for every $j = 1..m$, where $|\delta_j| < \delta$ for some $\delta > 0$, then

$$A_p(S) - \delta \leq A_p(T) \leq A_p(S) + \delta.$$

p-averages and approximation schemes for the p-Laplacian

For $p = 2$ we can rewrite the classical 5-points finite difference formula for the two dimensional Laplacian in terms of 2-average:

$$\begin{aligned} \Delta_2^G u(x_1, x_2) \quad & \approx \frac{1}{2h^2} [u(x_1 + h, x_2) + u(x_1, x_2 + h) \\ & + u(x_1 - h, x_2) + u(x_1, x_2 - h) - 4u(x_1, x_2)] = \\ & = \frac{2}{h^2} [A_2(C_h(\mathbf{x}, u)) - u(\mathbf{x})], \end{aligned}$$

where $C_h(\mathbf{x}, u)$ is the set of the four values of u in the adjacent nodes, that is

$$C_h(\mathbf{x}, u) = \{u(x_1 + h, x_2), u(x_1, x_2 + h), u(x_1 - h, x_2), u(x_1, x_2 - h)\}.$$

We could pick as $C_h(\mathbf{x}, u)$ even a larger set of values of u on the sphere of radius h, but since the Laplacian is a linear operator, this would not increase the accuracy.

p-averages and approximation schemes for the p-Laplacian

For $p = \infty$ ([Oberman, '04a]) :

$$\Delta_\infty^G u(\mathbf{x}) \approx \frac{2}{h^2} \left[A_\infty(C_h(\mathbf{x}, u)) - u(\mathbf{x})\right],$$

where now $C_h(\mathbf{x}, u)$ is a discrete set of values of u on the $B(\mathbf{x}, h)$, and the distribution and number of points on the sphere influences the accuracy of the approximation.

For $p = 1$ a similar approach has been proposed by [Oberman, '04b]

This suggests the following generalization to the game p-Laplacian $\forall p$:

$$\Delta_p^G u(\mathbf{x}) \approx \frac{2}{h^2} \left[A_p(C_h(\mathbf{x}, u)) - u(\mathbf{x})\right], \tag{11}$$

where $C_h(\mathbf{x}, u)$ would be a suitable discrete set of values of u on $B(\mathbf{x}, h)$.

The semi-Lagrangian scheme

We are then lead to the following approximation scheme for problem (8):

$$S(\rho, \mathbf{x}, u(\mathbf{x}), u) = 0 \ \text{ in } \overline{\Omega}, \tag{12}$$

where $\rho := (h, \Delta\theta)$ (with h spatial step and $\Delta\theta$ angular resolution), and $S : [0, 1) \times (0, \pi/2] \times \overline{\Omega} \times \mathbb{R} \times L^\infty(\overline{\Omega}) \longrightarrow \mathbb{R}$ defined as

$$S(\rho, \mathbf{x}, u(\mathbf{x}), u) = \begin{cases} -\frac{2}{\alpha^2 h^2}\left[A_p(C_h^{\Delta\theta}(\mathbf{x}, u; \alpha)) - u(\mathbf{x})\right] - f(\mathbf{x}) & \text{in } \Omega, \\ u(\mathbf{x}) - F(\mathbf{x}) & \text{on } \partial\Omega. \end{cases}$$

If $d_\Omega < \infty$ denotes the diameter of Ω, $\alpha = \alpha(\mathbf{x})$ is a dilation parameter such that $0 < \alpha(\mathbf{x}) \leq dist(\mathbf{x}, \partial\Omega) < d_\Omega$.
$C_h^{\Delta\theta}(\mathbf{x}, u; \alpha)$ is a suitably chosen discrete set of values of u, taken on the sphere $B(\mathbf{x}, h\alpha)$, associated to the angular resolution $\Delta\theta$.

Convergence results

- The scheme is monotone:
  if $u, v \in L^\infty(\overline{\Omega})$, $u(\mathbf{x}) \geq v(\mathbf{x})$ in $\overline{\Omega}$, then for all $p \geq 1$, $\rho \in [0,1) \times (0, \pi/2]$, $\mathbf{x} \in \overline{\Omega}$, $t \in \mathbb{R}$ it holds
  $$S(\rho, \mathbf{x}, t, u) \leq S(\rho, \mathbf{x}, t, v).$$
- For any $p \geq 2$ the scheme is consistent:
  for all $\mathbf{x} \in \Omega$ and $\phi \in C^\infty(\overline{\Omega})$, we have that
  $$\lim_{\rho \to 0} \frac{2}{\alpha^2 h^2} \left[A_p(C_h^{\Delta\theta}(\mathbf{x}, \phi; \alpha)) - \phi(\mathbf{x}) \right] = \begin{cases} \Delta_p^G \phi(\mathbf{x}) & \text{if } \nabla\phi(\mathbf{x}) \neq 0, \\ \Delta_2^G \phi(\mathbf{x}) & \text{if } \nabla\phi(\mathbf{x}) = 0. \end{cases}$$
- The scheme is stable (when $f \equiv 0$):
  $\forall h > 0, \Delta\theta > 0$, there exists a solution $u_\rho \in L^\infty(\overline{\Omega})$ of $S(\rho, x, t, u_\rho) = 0$ such that $\|u_\rho\|_\infty \leq \|F\|_\infty$.

Convergence follows by a result of Barles-Souganidis, '91

"Any monotone, stable and consistent approximation scheme to fully nonlinear second order elliptic or parabolic, possibly degenerate, PDE converges to the correct (viscosity) solution provided that there exists a comparison principle for the limiting equation."

Assume $f \equiv 0$, then there exists a unique bounded viscosity solution for problem (2), for which a comparison principle holds.
Since the scheme is monotone, stable and consistent (for $p \geq 2$), then it is *convergent*:

The solution u_ρ of (12) converges as $\rho \to 0$ to the unique viscosity solution of (8).

Numerical implementation

- We have introduced an approximation scheme which, at least under suitable hypotheses ($f \equiv 0$, $p \geq 2$) converges to the unique viscosity solution of the Dirichlet problem for the game p-laplacian
- In view of a practical implementation of the scheme we introduce a structured grid and an explicit time marching algorithm for the problem

$$u_t = \Delta_p^G u + f$$

- Since the points used by the set $C_h^{\Delta\theta}$ for the p-averages are not in general grid points, we have to introduce interpolation techniques which respect the monotonicity of the scheme
- We are able to prove that for $f \equiv 0$ and suitable initial conditions the iterates of our totally discrete scheme converge
- Numerical tests show that they converge to the right viscosity solution of the stationary problem

Space discretization and interpolation

- $h > 0$ space discretization step ($\{\mathbf{x}_j\}_{j=1}^N$ nodes on a h-uniform grid)
- $\Delta\theta = \frac{\pi}{2m}$ angular discretization step ($\rightarrow 4m$ points on the disk)
- $C_h^{\Delta\theta}(\mathbf{x}, u; \alpha)) = \{u(\mathbf{y}^i), i = 0, 1, .., 4m-1\}$ set for the p-average
  - $\mathbf{y}^i = \mathbf{x} + h\alpha r_i, \quad r_i = (\cos i\Delta\theta, \sin i\Delta\theta)$
  - $0 < \alpha^* < \alpha(x) \leq dist(\mathbf{x}, \partial\Omega)$ dilation parameter
- $\mathbf{y}^i$ not in general grid points $\rightarrow$ bilinear interpolation
- $\widehat{C}_h^{\Delta\theta}(\mathbf{x}_j, u; \alpha_j) = \{I[u](\mathbf{y}_j^i), i = 0, 1, .., 4m-1\}$
  - $I[u](\mathbf{y}) = ay_1y_2 + by_1 + cy_2 + d = \sum_{k=1}^4 u(\mathbf{x}_k)\lambda_k(\mathbf{y})$
  - $\mathbf{x}_k$ four vertices of the cell where $\mathbf{y}$ is
  - $\lambda_k(\mathbf{y})$ given functions dependings on the coordinates of $\mathbf{x}_k$
  - α_j may vary from point to point ($\rightarrow$ *multi-level circle stencil*)
    $\alpha_j(\mathbf{x}_j) = \beta \min(s, d_j/h) \quad (s \in \mathbb{N},\ 0 < \beta \leq 1,\ d_j = dist(\mathbf{x}_j, \partial\Omega))$

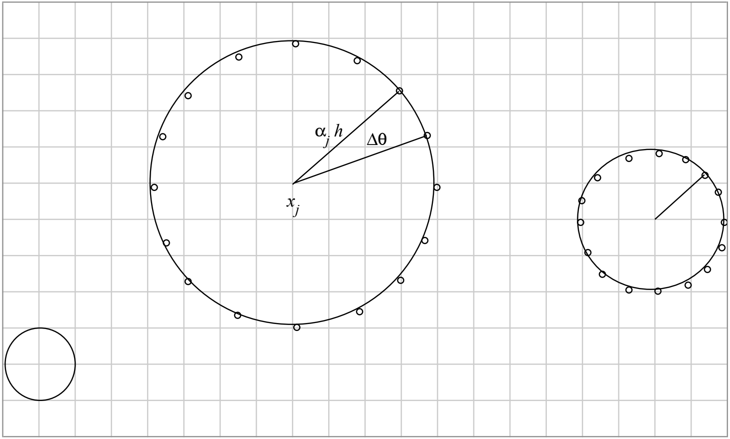

Figure: The 4-level circle stencil.

Remarks

- The choice of bilinear interpolation is essential to preserve the monotonicity of the scheme which is needed for the convergence proof. Quadratic interpolation has not this property.
  Bilinear interpolation satisfies:
  - $\min_D u(x) \leq I[u](x) \leq \max_D u(x)$
  - $I[u+\delta](x) = I[u] + \delta$

- With our choise of $\Delta\theta$ the number of equally distributed points on the sphere of radius $h\alpha_j$ is a multiple of 4. This means that if $\mathbf{r}$ is an admissible direction, so is its opposite $-\mathbf{r}$, as well as its orthogonal and its reflections with respect of each of the axes.

- The tests show that the multi-level circle strategy is able in general to speed up the convergence (reducing the number of iterations).

The practical scheme

On the previous grid, given an initial condition $\mathbf{u}^0 \in \mathbb{R}^N$, we implement the explicit time marching scheme:

$$\mathbf{u}^{n+1} = T_\rho(\mathbf{u}^n) := \mathbf{u}^n - \Delta t \; S(\rho, \mathbf{y}, \mathbf{u}^n(\mathbf{y}), \mathbf{u}^n),$$

which is consistent, because the scheme S is consistent in the stationary case. Taking care of the interpolation, we get

$$u_j^{n+1} = \begin{cases} u_j^n + \dfrac{2\,\Delta t}{\alpha_j^2\, h^2} \left[A_p(\widehat{C}_h^{\Delta\theta}(\mathbf{x}_j, \mathbf{u}^n; \alpha_j)) - \; u_j^n \right] + \Delta t \; f(\mathbf{x}_j) & \mathbf{x}_j \in \Omega, \\ \\ F(\mathbf{x}_j) & \mathbf{x}_j \in \partial\Omega; \end{cases}$$

until a stopping criterion is satisfied ($\max_j |u_j^{n+1} - u_j^n| \le \varepsilon$).

A simpler variant

If we choose parameters s.t. $\frac{2\Delta t}{\alpha_j^2 h^2} = 1$ the scheme simplifies as

$$u_j^{n+1} = \begin{cases} A_p(\widehat{C}_h^{\Delta\theta}(\mathbf{x}_j, \mathbf{u}^n; \alpha_j)) + \Delta t\, f(\mathbf{x}_j) & \text{if } \mathbf{x}_j \in \Omega, \\ F(\mathbf{x}_j) & \text{if } \mathbf{x}_j \in \partial\Omega; \end{cases}$$

Remark. The p-averages can be computed by any standard method for minimization of convex function. We used the Newton Bracketing method, but this part can be optimized to improve the speed of calculations.

The Newton-Bracketing method

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that its minimum value f_{min} is attained, and is contained in the bracket $[L, U = f(x)]$ for some point x such that $f'(x) \neq 0$. Then the iterative method generates a sequence of nested brackets shrinking to a point:

1. Stopping rule: If $U - L < \varepsilon$ stop with x as a solution.
2. Select a value $M := \alpha U + (1 - \alpha)L$ for some $0 < \alpha < 1$.
3. Do one Newton iteration $x_+ = x - \frac{f(x)-M}{f'(x)}$.
4. Case 1: If $f(x_+) < f(x)$ then update $U : U_+ := f(x_+)$ and leave $L_+ := L$. Go to 1.
5. Case 2: If $f(x_+) \geq f(x)$ then update $L : L_+ := M$ and leave $U_+ := U$, $x_+ := x$. Go to 1.

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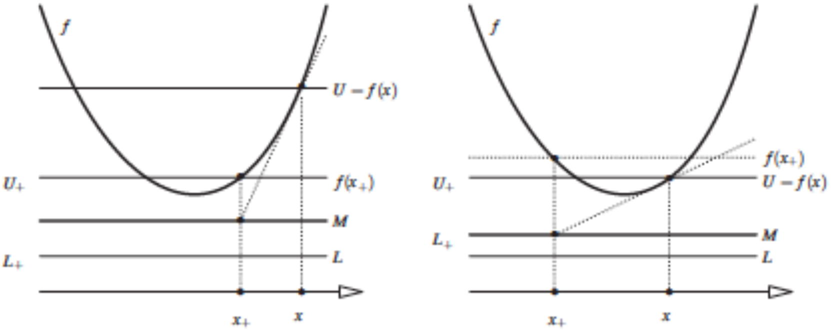


(a) Case 1: $f(x_+) < f(x)$ (b) Case 2: $f(x_+) \geq f(x)$

Fig. 4.1 Illustration of the 2 cases of the NB method

Convergence of the iterations

- Let $\dfrac{2\,\Delta t}{\alpha_*^2\, h^2} \leq 1$. Assume $f \equiv 0$, then for $n \geq 1$ it holds

$$\sup_{j=1..N} |u_j^n| \leq \sup_{j=1..N} |u_j^{n-1}| \leq \sup_{j=1..N} |u_j^0| \quad \text{(stability)}$$

- Let $\dfrac{2\,\Delta t}{\alpha_*^2\, h^2} \leq 1$ and $\mathbf{u}^0$ given by

$$u_j^0 = \begin{cases} \min\limits_{\partial\Omega} F & \text{if } \mathbf{x}_j \in \Omega, \\ F(\mathbf{x}_j) & \text{if } \mathbf{x}_j \in \partial\Omega. \end{cases}$$

  Then for $n \geq 1$ the iterations generated by the scheme verify

$$u_j^n \geq u_j^{n-1} \text{ for any } j = 1..N, \text{ and } n \geq 1 \quad \text{(pointwise monotonicity)}$$

- Then for $f \equiv 0$ and an appropriately chosen initial condition the scheme is pointwise convergent.

Example 1: Aronsson function

$$\Omega = (-1,1)^2, \quad p = \infty, \quad f \equiv 0, \quad F(x,y) = |x|^{4/3} - |y|^{4/3}$$

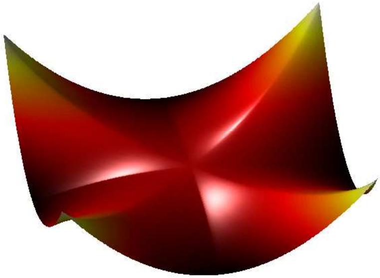
Then $u = |x|^{4/3} - |y|^{4/3}$ is the exact solution of the problem (an example of an absolute minimizer which is not twice differentiable). The scheme gives the following results (N^2= nodes, d= directions) when a 2-level iterations is used (with $\beta = 0.99$):

d	$N = 41$	$N = 81$	$N = 161$	$N = 241$	$N = 401$
4	0.1105 (250)	0.0765 (448)	0.0373 (584)	0.0225 (589)	0.0122 (621)
8	0.0274 (80)	0.0182 (161)	0.0084 (214)	0.0069 (190)	0.0048 (188)
16	0.0084 (54)	0.0070 (75)	0.0043 (105)	0.0033 (108)	0.0023 (112)
24	0.0088 (57)	0.0081 (73)	0.0050 (91)	0.0035 (103)	0.0024 (107)

Table: L^∞- errors and iterations (in parentheses) for test on Aronsson function

Example 2

$$\Omega = (-1,1)^2, \quad p = \infty, \quad f = 0, \quad F(x,y) = |x|^2\,|y|^2$$



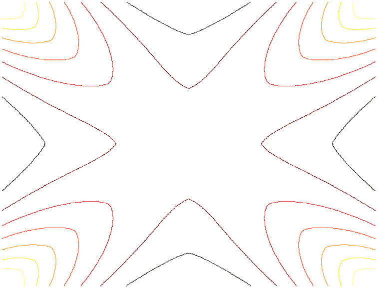


Figure: $N = 401^2$, $d = 24$, 2-lev ($\beta = 0.99$).

Example 3

$$\Omega = (-1,1)^2, \quad p = \infty, \quad f = 0, \quad F(x,y) = x^3 - 3xy^2$$

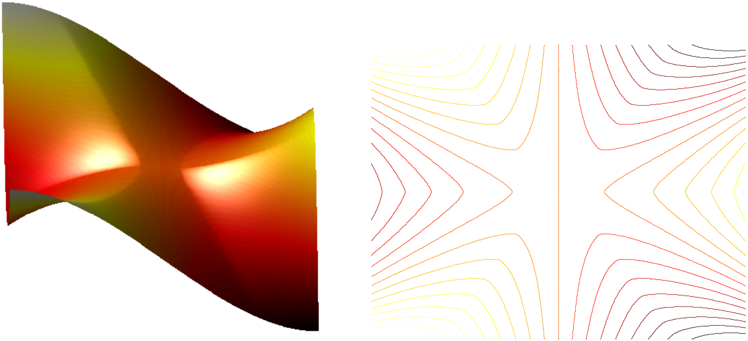


Figure: $N = 401^2$, $d = 24$, 2-lev ($\beta = 0.99$).

Example 4

$\Omega = (-1, 1)^2$, $p = \infty$, $f = 0$, F characteristic function of point $(1, 0)$

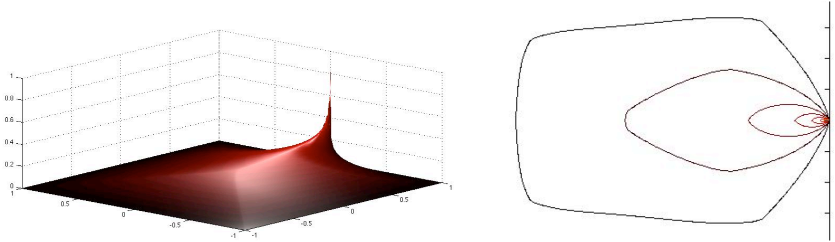


Figure: $N = 401^2$, $d = 24$, 4-lev ($\beta = 0.99$).

Example 5

$$\Omega = (-1,1)^2, \quad p \geq 2, \quad f = 1, \quad F(x,y) = (1-x^2-y^2)/2$$

The exact solution is known: $u(x,y) = (1-x^2-y^2)/2$, the function which solves the problem with $F = 0$ on the unit sphere.

N (lev)	$d = 16$	$d = 24$
21 (2)	0.0634 (163)	0.0617 (180)
21 (4)	0.0241 (50)	0.0192 (107)
41 (4)	0.0201 (213)	0.0191 (163)

Table: L^∞-errors and iterations for $p = 5$, $\beta = 0.9$.

N (lev)	$d = 16$	$d = 24$
21 (2)	0.0590 (249)	0.0563 (248)
21 (4)	0.0211 (80)	0.0185 (77)
41 (4)	0.0192 (272)	0.0156 (272)

Table: L^∞-errors and iterations for $p = \infty$, $\beta = 0.9$.

Example 6

$$\Omega = (-2,2) \times (-1,1), \quad p = \infty, \quad f \equiv 1, \quad F \equiv 0$$

The exact solution is known only in part of the domain:
$u(x,y) = (1-y^2)/2$ when $|x| \leq 1$

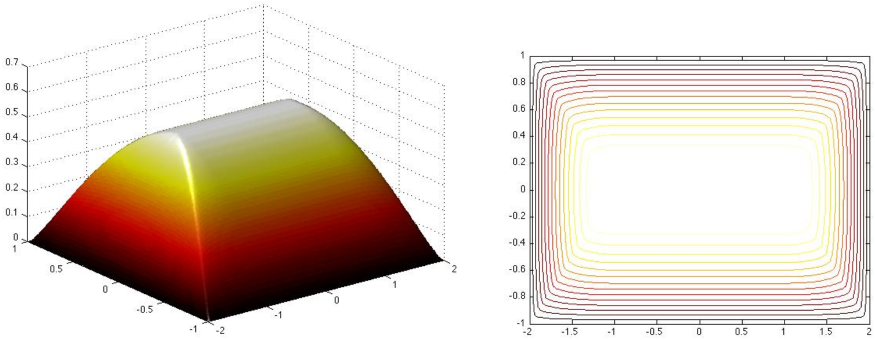


Figure: Surface and contour plots for $N = 201 \times 101$, $d = 16$, 4-lev ($\beta = 0.8$).

Thank you