New models and techniques for computing optical flows and digital inpainting

Alfio Borzì

Università degli Studi del Sannio
Dipartimento e Facoltà di Ingegneria, Benevento, Italia

Institute for Mathematics and Scientific Computing
Karl-Franzens-University, Graz, Austria

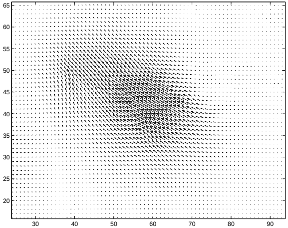
Content

New formulation for the optical flow problem and the digital inpainting problem.

For determining optical flows, an optimization approach is presented where the velocity field components are interpreted as control functions of the optimal control problem of tracking a sequence of given frames.

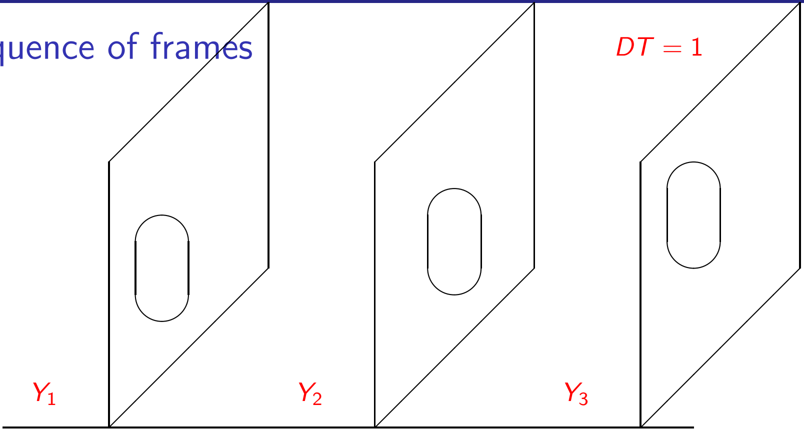
For digital inpainting, a new approach based on the solution of a Ginzburg–Landau equation is discussed.

Optical flow



Optical flow: The field of apparent velocities of movement of brightness points in a sequence of images.
Assumptions: Objects represented in the image are flat surfaces, are uniformly illuminated, and reflectance varies smoothly.
Applications: Based on information about spatial arrangement of objects and the rate of change of this arrangement.

Sequence of frames



Each frame is composed of $L \times L$ pixels ($DX = 1$).

Y represents grey or color values. Y may be affected by noise.

Università degli Studi del Sannio

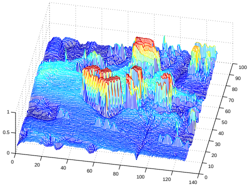
Formulation of the problem

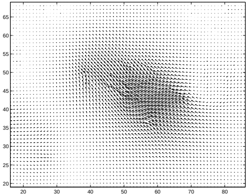
Find $\vec{w} = (u, v)$ such that $I_t + \vec{w} \cdot \vec{\nabla} I = 0$ is satisfied and $I(\cdot, t_k) \approx Y_k$.

Difficulties

- There are two unknown components of the optical flow.
- Needs of auxiliary constraint or regularization.
- Inverse problem associated to the hyperbolic OFC equation.
- Given Y, extract approximation to the spatio-temporal derivatives, (Y_x, Y_y, Y_t).

Input and output





The Horn & Schunck Method

OFC equation + global smoothness term

Minimizing:

$$\int_D [(I_t + \vec{w} \cdot \vec{\nabla} I)^2 + \lambda^2 (|\nabla u|^2 + |\nabla v|^2)] d\mathbf{x}, \qquad (1)$$

A minimum of (1) satisfies necessarily the Euler equations:

$$\begin{aligned} \lambda^2 \Delta u - I_x (I_t + uI_x + vI_y) &= 0, \\ \lambda^2 \Delta v - I_y (I_t + uI_x + vI_y) &= 0, \end{aligned}$$

where (I_t, I_x, I_y) are obtained by finite differentiation of the data Y_k.

Optimal control framework

Observation: (u, v) determine the transformation of an image frame to the next.

Idea: Take them as control functions of the following optimal control problem:

Find $\vec{w}$ and I such that

$$\begin{cases} I_t + \vec{w} \cdot \nabla I &= 0, \text{ in } Q = \Omega \times (0, T), \\ I(\cdot, 0) &= Y_1, \end{cases}$$

and minimize the cost functional

$$\begin{aligned} J(I, \vec{w}) &= \frac{1}{2} \sum_{k=1}^{N} \int_{\Omega} |I(x, y, t_k) - Y_k|^2 d\Omega + \frac{\alpha}{2} \int_Q \Phi(|\frac{\partial \vec{w}}{\partial t}|^2) dq \\ &+ \frac{\beta}{2} \int_Q \Psi(|\vec{\nabla} u|^2 + |\vec{\nabla} v|^2) dq + \frac{\gamma}{2} \int_Q |\vec{\nabla} \cdot \vec{w}|^2 dq, \end{aligned}$$

where α, β, and γ are the weights of the cost of the control.

Cost functional $J(I, \vec{w})$

$I(\cdot, t_k, \vec{w})$ approximates Y_k: $\frac{1}{2}\sum_{k=1}^{N}\int_\Omega |I(x,y,t_k) - Y_k|^2 d\Omega$

$\vec{w}$ is smooth with respect to t: $\frac{\alpha}{2}\int_Q \Phi(|\frac{\partial \vec{w}}{\partial t}|^2)dq, \qquad \Phi(s) = s$

$\vec{w}$ piecewise smooth in the spatial variables: $\frac{\beta}{2}\int_Q \Psi(|\vec{\nabla} u|^2 + |\vec{\nabla} v|^2)dq,$

$$\Psi(s) = \begin{cases} 2\sqrt{s} & \text{for } s \in [0, \delta), \\ s + c_1 & \text{for } s \in [\delta, \delta'], \\ 2\sqrt{s} + c_2 & \text{for } s \in (\delta', \infty). \end{cases}$$

Filling-in: $\frac{\gamma}{2}\int_Q |\vec{\nabla} \cdot \vec{w}|^2 dq$

Optimality system

The state equation (constraint) evolving forward:

$$I_t + \vec{w} \cdot \nabla I = 0.$$

The adjoint equation evolving backwards:

$$\begin{aligned} p_t + \nabla \cdot (\vec{w} p) &= 0, \text{ on } t \in (t_{k-1}, t_k), \\ p(\cdot, t_k^+) - p(\cdot, t_k^-) &= I(\cdot, t_k) - Y_k, \quad t = t_k, \end{aligned}$$

for $k = 2, \ldots N - 1$.
Two (space-time) elliptic control equations:

$$\begin{aligned} \alpha \frac{\partial^2 u}{\partial t^2} + \beta \nabla \cdot [\Psi'(|\nabla \vec{w}|^2) \nabla u] + \gamma \frac{\partial}{\partial x} (\nabla \cdot \vec{w}) &= p \frac{\partial I}{\partial x}, \\ \alpha \frac{\partial^2 v}{\partial t^2} + \beta \nabla \cdot [\Psi'(|\nabla \vec{w}|^2) \nabla v] + \gamma \frac{\partial}{\partial y} (\nabla \cdot \vec{w}) &= p \frac{\partial I}{\partial y}, \end{aligned}$$

where $|\nabla \vec{w}|^2 = |\nabla u|^2 + |\nabla v|^2$.

Optimality system (continue)

Initial conditions and terminal conditions

For the (forward) optical flow equation:

$$I(x, y, t)|_{t=0} = Y_1(x, y).$$

For the (backward) adjoint optical flow equation:

$$p(x, y, t)|_{t=T} = -(I(x, y, T) - Y_N(x, y)).$$

For the control equations:

$$\frac{\partial \vec{w}}{\partial t} = 0, \text{ at } t = 0 \text{ and } t = T; \quad \vec{w} = 0, \text{ on } \partial\Omega \times (0, T).$$

Numerical schemes: Explicit Second-Order TVD Scheme

The adjoint equation. $p_{t'} + \vec{\nabla}\cdot(-\vec{w}p) = -\sum_{k=2}^{N-1}\delta(t,t_k)(I - Y_k)$

$$\begin{aligned}\frac{dp_i}{dt'} = & - \frac{1}{h}[1 + \frac{1}{2}\chi(r^{+}_{i-1/2}) - \frac{1}{2}\frac{\chi(r^{+}_{i-3/2})}{r^{+}_{i-3/2}}](-u)^{+}_{i-1/2}(p_i - p_{i-1}) \\ & - \frac{1}{h}[1 + \frac{1}{2}\chi(r^{-}_{i+1/2}) - \frac{1}{2}\frac{\chi(r^{-}_{i+3/2})}{r^{-}_{i+3/2}}](-u)^{-}_{i+1/2}(p_{i+1} - p_i)\end{aligned}$$

with Superbee limiter.
Delta impulses: splitting technique at t_k

$$p(\cdot,t_\kappa) = p(\cdot,t_\kappa^{+}) - (I(\cdot,t_k) - Y_k) \text{ for } t_{\kappa+1} = t_k,$$

$p(\cdot,t_\kappa^{+})$: by solving the adjoint equation with init. cond. $p(\cdot,t_{\kappa+1})$.

Università degli Studi del Sannio

Numerical schemes: Multigrid Algorithm

Discretized elliptic equation (e.g., u component):

$$
\begin{aligned}
&\alpha \frac{u_{i,j,\kappa+1}-2u_{i,j,\kappa}+u_{i,j,\kappa-1}}{\tau^2} + \beta\{\nabla^h \cdot [\Psi'(|\nabla^h \vec{w}^h|^2)\nabla^h u^h]\}_{i,j,\kappa} \\
&+\gamma \frac{u_{i+1,j,\kappa}-2u_{i,j,\kappa}+u_{i-1,j,\kappa}}{h^2} \\
&= [p\frac{\partial I}{\partial x}]_{i,j,\kappa} - \gamma \frac{v_{i+1,j+1,\kappa}-v_{i+1,j-1,\kappa}-v_{i-1,j+1,\kappa}+v_{i-1,j-1,\kappa}}{4h^2}.
\end{aligned}
$$

Multigrid FAS method for solving $A^h(\phi^h) = f^h$.

1. Apply ν_1 smoothing steps: $\phi^h = S^{\nu_1}(\phi^h, f^h)$.
2. Transfer the approximate solution: $\phi^H = \hat{I}_h^H \phi^h$.
3. Compute the right hand side of the FAS equation:

$$f^H = I_h^H f^h + [A^H(\phi^H) - I_h^H A^h(\phi^h)].$$

4. Apply γ times the FAS scheme to $A^H(\hat{\phi}^H) = f^H$.
5. Use coarse level correction: $\phi^h = \phi^h + I_H^h(\hat{\phi}^H - \phi^H)$.
6. Apply ν_2 smoothing steps: $\phi^h = S^{\nu_2}(\phi^h, f^h)$.

Solution Process

Segregation loop for solving the optimal control problem.

- Apply the Horn & Schunck method for starting approximation to the optical flow.

  1. Solve the optical flow constraint equation to obtain I.
  2. Solve the adjoint optical flow constraint equation to obtain p.
  3. Update the right-hand sides of the elliptic system, compute $p\vec{\nabla} I$.
  4. Apply a few V-cycles of multigrid to solve the control equations, obtain $\vec{w}$.
  5. Go to 1 and repeat I_{loop} times.

Result variables

Consider $\vec{w} = (u, v, 1)$ in units of (pixel,pixel,frame).

Direction vector: $\hat{w} = \frac{1}{\sqrt{1+u^2+v^2}}(u, v, 1)^T$.

Orientation error: $\psi^E_{i,j,\kappa} = \arccos(\hat{w}^c_{i,j,\kappa} \cdot \hat{w}^e_{i,j,\kappa})$

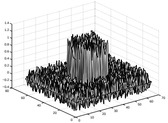
Mean orientation error:

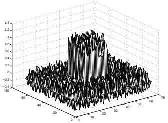
$$\bar{\psi} = \frac{1}{KL^2} \sum_{\kappa=1}^{K} \sum_{i,j=1}^{L} \psi^E_{i,j,\kappa}.$$

Tracking error:

$$||I - Y||^2 = \sum_{\kappa=1}^{K} \sum_{i,j=1}^{L} (I^{t_k}_{i,j} - Y(x_i, y_j, t_k))^2.$$

Two synthetic images of moving square (with 30% noise) $(u_c, v_c) = (1.5, 2)$

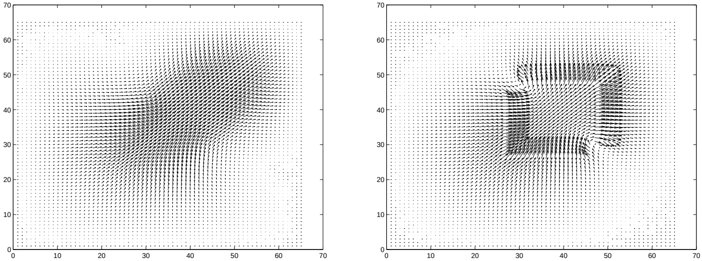




Dependence on α; $\beta = 0.25$, and $\gamma = 0.1$.					
α	$\lvert u\rvert_{max},\ \lvert v\rvert_{max}$	$\bar{\psi}$	$\lvert\lvert I - Y\rvert\rvert^2$	$\sum cost$	$\lvert\lvert div(w)\rvert\rvert^2$
0.5	1.59, 2.08	44.1	107.4	20.1	8.5(-1)
1	1.78, 2.27	45.7	104.7	45.6	1.6
5	2.08, 2.34	48.4	98.5	145.5	3.9

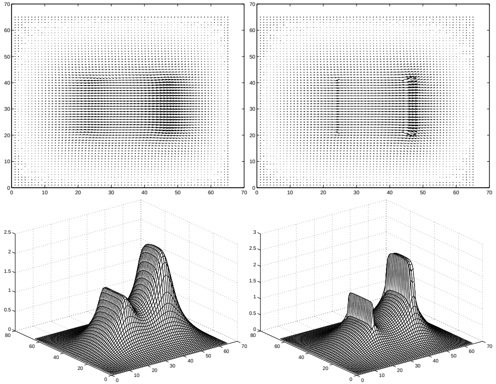
Dependence on γ; $\alpha = 1.0$, and $\beta = 0.25$.					
γ	$\lvert u\rvert_{max},\ \lvert v\rvert_{max}$	$\bar{\psi}$	$\lvert\lvert I - Y\rvert\rvert^2$	$\sum cost$	$\lvert\lvert div(w)\rvert\rvert^2$
0	1.86, 2.34	46.5	102.4	65.1	0
0.5	1.53, 2.03	43.5	111.6	16.5	1.0
1	**1.39, 1.82**	**41.8**	**117.9**	**7.9**	**6.9(-1)**
2	1.31, 1.58	39.8	127.6	4.0	4.3(-1)
H & S	2.18, 2.28	49.9			

Fast moving objects



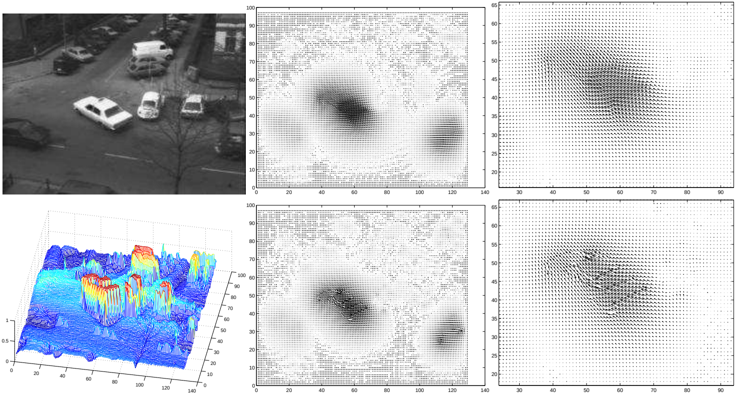
Test case: $(u_c, v_c) = (5, 5)$. Optical flow obtained with optimal control method (left) and with Horn & Schunck scheme (right).

Moving and dilating objects



Optical flow with optimal control (left) and with Horn & Schunck (right); corresponding u component (bottom); the v component is approx. zero.

Real images



Optical flow with optimal control (top) and with Horn & Schunck (bottom)

Convergence results

Well-posedness of the iterative algorithm for the case $\Psi(s) = s$:

Theorem 1 Suppose that $I(0) = Y_1 \in H^2_{per}(\Omega)$, $Y_k \in H^1_{per}(\Omega) \cap W^{1,q}(\Omega)$ for some $q \in (2, \infty]$ and all $k = 2, \ldots, N$, and that $\Psi(s) = s$. Then, the iteration map F defined by the segregation loop $\vec{w}^{new} = F(\vec{w}^{old})$ is well-defined provided $\gamma > 0$ is sufficiently small.

Let W be the restoring energy cost defined by

$$W(\vec{w}) = \int_0^T \int_\Omega \left(\frac{\alpha}{2} \, \left| \frac{\partial \vec{w}}{\partial t} \right|^2 + \frac{\beta}{2} \, (|\nabla u|^2 + |\nabla v|^2) + \frac{\gamma}{2} \, |\nabla \cdot \vec{w}|^2 \right) dxdt.$$

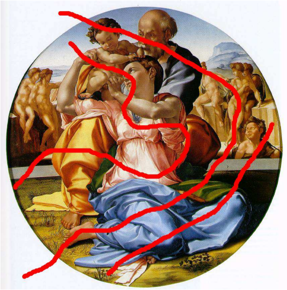
Theorem 2 Suppose that the hypotheses of Theorem 1 hold and denote by $(I^n, \vec{w}^n)$ and $(I^{n+1}, \vec{w}^{n+1})$ two consecutive iterates of the proposed algorithm. Then we have

$$J(I^{n+1}, \vec{w}^{n+1}) - J(I^n, \vec{w}^n) = -W(\vec{w}^{n+1} - \vec{w}^n)$$

$$- \tfrac{1}{2} \textstyle\sum_{k=2}^N \int_\Omega |I^{n+1}(t_k) - I^n(t_k)|^2 \, dx$$

$$+ \textstyle\int_0^T \int_\Omega (p^{n+1} - p^n) \, (\vec{w}^{n+1} - \vec{w}^n) \cdot \nabla I^n \, dxdt.$$

Digital inpainting



Digital inpainting is the process of restoration of missing image data performed by computer programs (requiring a user only to mark inpainting domains in a digitized image).
Digital inpainting has several applications in photography such as scratch removal or retouching.

Digital inpainting and the Ginzburg–Landau equation

Solutions of the real valued Ginzburg–Landau equation develop areas with values ± 1, which are separated by interfaces of minimal area.

We focus on inpainting of gray–valued or color images. For this purpose we use the complex valued Ginzburg–Landau equation.

The Ginzburg–Landau Equation (GLE)

Ginzburg & Landau derived the following approximation for the thermodynamic energy related to superconductors:

$$F(u, \nabla u) := \frac{1}{2} \int_\Omega \underbrace{|-i\nabla u|^2}_{\text{kinetic term}} + \underbrace{\alpha |u|^2 + \frac{\beta}{2}|u|^4}_{\text{potential term}}$$

where $u : \Omega \to \mathbb{C}$ is called the *order function*, and $\alpha < 0$ and $\beta > 0$ are physical constants.

The state of minimal energy satisfies the Euler equation $\delta F(u, \nabla u)/\delta u = 0$. This is the stationary Ginzburg–Landau equation

$$\Delta u + \frac{1}{\varepsilon^2}\left(1 - |u|^2\right) u = 0\,.$$

Where the minima of the potential term function are attained at the sphere $|u| = 1$ choosing $\alpha = -\frac{1}{\varepsilon^2}$ and $\beta = \frac{1}{\varepsilon^2}$.

The GLE setting for inpainting

Let D be the domain of the image, usually a rectangular subset of $\mathbb{R}^2$.
The inpainting domain is denoted by $\Omega \subset D$.

Let $u^0 : D \to \mathbb{C}$ be defined by the given image. We define $\Re e(u^0) =: \overline{u}^0 : D \to [-1, 1]$ be the gray–value intensity of an image scaled to the interval $[-1, 1]$; with values -1 (white) and 1 (black).
Further, we set $\Im m(u^0) = \sqrt{1 - \left(\overline{u}^0\right)^2}$ such that $|u^0(x)| = 1$ for all $x \in D$.

Let u be the solution to the GLE with Dirichlet boundary condition $u|_{\partial\Omega} = u^0|_{\partial\Omega}$. $\Re e(u)$ is the inpainting function.
The GLE can be generalized to RGB color images where $u : D \to \mathbb{C}^3$. In this case we replace $|\cdot|$ with

$$\|u(x)\| := \max\{|u^1(x)|, |u^2(x)|, |u^3(x)|\}$$

The corresponding GLE cannot be derived from a variational principle.

GLE Solution by time-stepping

For the numerical solution of the GLE, we use a relaxation procedure corresponding to the time evolution of

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2}\left(1 - |u|^2\right) u$$

towards a stationary state.

We consider θ–schemes defined as follows: For $\theta \in [0,1]$

$$\frac{u^{(k+1)} - u^{(k)}}{\delta_t} = \theta\left(\Delta_h u^{(k+1)} + \frac{1}{\varepsilon^2}\left(1 - |u^{(k+1)}|^2\right) u^{(k+1)}\right) + (1-\theta)\left(\Delta_h u^{(k)} + \frac{1}{\varepsilon^2}\left(1 - |u^{(k)}|^2\right) u^{(k)}\right)$$

We also consider implicit–explicit (IMEX) schemes

$$\frac{u^{(k+1)} - u^{(k)}}{\delta_t} = \Delta_h u^{(k+1)} + \frac{1}{\varepsilon^2}\left(1 - |u^{(k)}|^2\right) u^{(k)}$$

Time-stepping as a minimization procedure

Define $\lambda(v,w) = \frac{1}{\varepsilon^2}(1 - v^2 - w^2)$ and choose $\theta = 0$.
The solution of the discretized stationary GLE corresponds to the minimum of the discrete functional

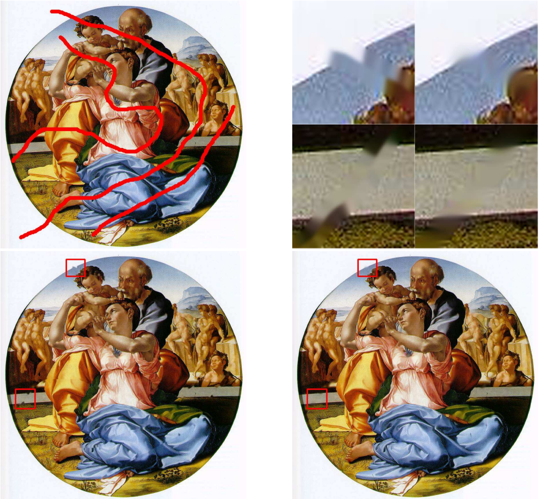
$$J_h(u) = \frac{1}{2}\left(\sum_{d\in\{x,y\}} \|\partial_d^- v\|_2^2 + \sum_{d\in\{x,y\}} \|\partial_d^- w\|_2^2 \right) + \frac{\epsilon^2}{4}\left(\lambda(v,w),\lambda(v,w)\right).$$

The gradient of $J_h(u)$ is given by

$$J_h(u)' = -(\Delta_h u + \frac{1}{\varepsilon^2}\left(1 - |u|^2\right) u)$$

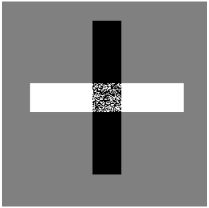
Explicit time step as a minimization step with step length δ_t

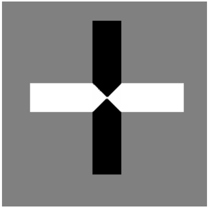
$$u^{(k+1)} = u^{(k)} - \delta_t\, J_h(u^{(k)})' \qquad \text{where} \qquad \delta_t \left(\frac{1}{h^2} + \frac{2}{\varepsilon^2} \right) \le 1.$$

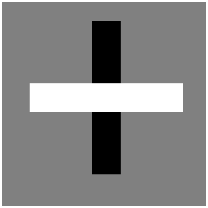


The painting "Holy Family" from Michelangelo with scratches (*top left*). The inpainted image with plain Ginzburg–Landau algorithm (*bottom left*). The inpainted image with the same algorithm interleaved with some steps of coherence enhancing diffusion (*bottom right*). Detailed views of the red framed parts are compared (*top*

Ambiguous example .....

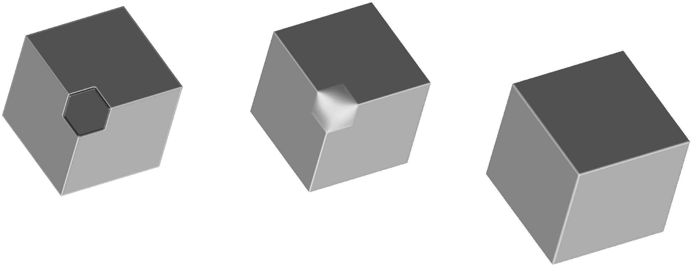






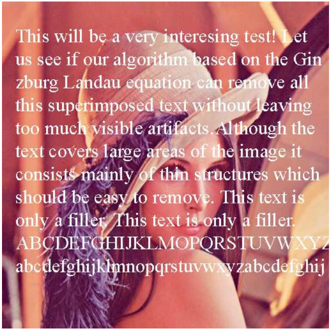
The noisy area should be inpainted (*first picture*), inpainting with the Ginzburg–Landau algorithm (*second picture*), inpainting via level set algorithm (*third picture*).

Three Dimensional Inpainting



A corner of the cube was manually cut out (*left*). The completion attained with a linear diffusion approach (*middle*). With the Ginzburg–Landau equation a perfect corner is achieved (*right*)

Thank you



Thanks for your attention !!