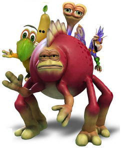
Continuous Models in Evolutionary Game Theory



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January 2012

Topics

- Mathematical models for population dynamics and evolution, based on game theory
- Models that are intermediate between the discrete approach of cellular automata and the reaction-diffusion models

Plan

- A continuous mixed strategies model
- A hyperbolic model $\rightarrowtail$ Analytical properties
- A hyperbolic model $\rightarrowtail$ Numerical simulations

Game theory

Game theory analyzes the interactions between two or more players.
The basic concepts of game theory are

- **Player's strategy**
- **Player's payoff**
- **Payoff matrix**

- **Player's strategy**
  It is a complete plan of action for whatever situation might arise and fully determines the player's behaviour.

Pure Strategy	Mixed Strategy
It defines a specific action that a player will follow in every possible situation	It is an assignment of a probability to each pure strategies

- **Player's strategy**
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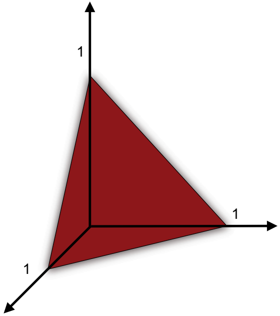
Pure Strategy
It defines a specific action that a player will follow in every possible situation

Mixed Strategy
It is an assignment of a probability to each pure strategies

Consider a game with N pure strategies $S_1, \ldots, S_N$.
A mixed strategy consists in playing the pure strategies with some probabilities q_1 to q_N, with $q_i \geq 0$ and $\sum_{i=1}^{N} q_i = 1$.
A strategy corresponds to a point $\mathbf{q}$ in the simplex

$$(1) \qquad \mathcal{S}_N := \{\mathbf{q} = (q_1, \ldots, q_N) \in \mathbb{R}^N : q_i \geq 0 \text{ and } \sum_{i=1}^{N} q_i = 1\}.$$

The corners of the simplex are the standard unit vectors $\mathbf{e}_i$ with the i-th component is 1 and all others are 0 and correspond to the N pure strategies S_i, $i = 1, \ldots, N$.



Game theory

- **Player's payoff** It represents the motivations of a player. A player, in a given game with other players, has to decide between different strategies in order to maximise the payoff, which depends on the strategies of co-players.
- **Payoff matrix** It expresses the relationships between players: its entries represent the payoffs that each player has in the game.

Game theory

The Prisoner's dilemma Game

T > R > P > S
2R > T+S

		Player 2: Cooperate	Player 2: Defect
Player 1	Cooperate	R, R	S, T
	Defect	T, S	P, P

It is a game with two players and two pure strategies. Players can cooperate or defect.

Consider a game with N players and a set of strategies available. Denote by s_i the strategy chosen by the i-th player and by $U_i(s_1, s_2, \ldots, s_i, \ldots, , s_n)$ the i-th player's payoff.
A **Nash Equilibrium** is a set of strategies s_1^ϵ, s_2^ϵ, ..., s_N^ϵ such that

$$U_i(s_1^\epsilon, s_2^\epsilon, \ldots, s_i^\epsilon, \ldots, s_N^\epsilon) \geq U_i(s_1^\epsilon, s_2^\epsilon, \ldots, s_i, \ldots, s_N^\epsilon),$$

for all i and for all strategy s_i chosen by the i-th player.
Consider the general payoff matrix between two strategies S_1 and S_2

$$\begin{array}{cc} & \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{array}{c} S_1 \\ S_2 \end{array} & \left(\begin{array}{cc} a & b \\ c & d \end{array} \right). \end{array} \tag{2}$$

We have the following criteria:

- S_1 is a Nash equilibrium if $a \geq c$
- S_2 is a Nash equilibrium if $d \geq b$.

Evolutionary dynamics is based on the use of ideas coming from game theory in the investigation of population dynamics:

- An entire population is a player in **the game of life**
- The payoff is interpreted as a **fitness**
- The success in the game is interpreted as **reproductive success**
- The strategies with high payoff reproduce faster; the strategies that do poorly are outcompeted. This means **natural selection**.

The replicator dynamics

The replicator equation describes the evolution over time of the frequencies of strategies in a population and were introduced by Taylor and Jonker in 1978. Assume that the population is divided into r types (strategies) E_1 to E_r with frequencies x_1 to x_r. The **replicator equations** read

$$\dot{x}_i = x_i[f_i(\mathbf{x}) - \Phi(\mathbf{x})], \qquad i = 1, \ldots, r \tag{3}$$

where

- x_i is the frequency of ith strategy
- $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{S}_r$ is the composition of the population
- $A \in \mathbb{R}^{r \times r}$ is the payoff matrix
- $f_i(\mathbf{x}) = (A\mathbf{x})_i$ is the fitness of the ith strategy
- $\Phi(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$ is the average fitness of population.

A continuous mixed strategies model

Joint work with R. Natalini and L. Pareschi [BNP11].
Consider a game with N pure strategies S_1 to S_N and assume that players can use mixed strategies. Denote by $f(t, \mathbf{q})$ the density of population adopting the $\mathbf{q}$ strategy at time t; the evolution in time of f, due to the dynamics of the game, is driven by

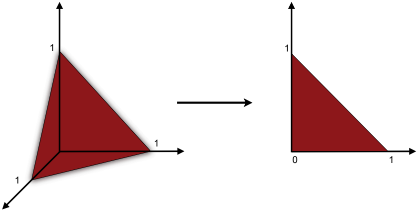
$$\partial_t f(t,\mathbf{q}) = f(t,\mathbf{q}) \left(\int_{\mathcal{S}_N} A(\mathbf{q},\mathbf{q}^*) f(t,\mathbf{q}^*)\, d\mathbf{q}^* - \phi(f) \right), \tag{4}$$

where

- $\int_{\mathcal{S}_N} A(\mathbf{q},\mathbf{q}^*) f(t,\mathbf{q}^*)\, d\mathbf{q}^*$ is the payoff of the strategy $\mathbf{q}$ against all the others strategies
- $A(\mathbf{q},\mathbf{q}^*) = \mathbf{q} \cdot A\mathbf{q}^* = \sum_{i,j=1}^{N} a_{ij} q_i q_j^*$. is the interacting kernel between the $\mathbf{q}$-strategist and the $\mathbf{q}^*$-strategist
- $\phi(f) := \int_{\mathcal{S}_N} \int_{\mathcal{S}_N} f(t,\mathbf{q})\, A(\mathbf{q},\mathbf{q}^*)\, f(t,\mathbf{q}^*)\, d\mathbf{q}^*\, d\mathbf{q}$ represents the average payoff of the population.

A continuous mixed strategies model

Since $\sum_{i=1}^{N} q_i = 1$, we can reduce the number of variables, considering $q_N = 1 - \sum_{i=1}^{N-1} q_i$ and obtaining the $(N-1)$ - dimensional model (4), on the simplex



$$(5) \qquad \mathcal{T}_{N-1} := \{\mathbf{p} = (p_1, p_2, \ldots, p_{N-1}) \in \mathbb{R}^{N-1} \,|\, p_i \geq 0\,, \sum_{i=1}^{N-1} p_i \leq 1\}.$$

We consider an initial condition $f(0, \mathbf{p}) = f_0(\mathbf{p})$.

- If $f_0(\mathbf{p}) \geq 0$, then $f \geq 0 \quad \forall t > 0$
- if $f_0(\tilde{\mathbf{p}}) = 0$ for some $\tilde{\mathbf{p}}$, then $f(t, \tilde{\mathbf{p}}) = 0 \quad \forall t > 0$
- if $\int_{\mathcal{T}_{N-1}} f_0(\mathbf{p}) d\mathbf{p} = 1$, then $\int_{\mathcal{T}_{N-1}} f(t, \mathbf{p}) d\mathbf{p} = 1, \quad \forall\, t > 0.$

A continuous mixed strategies model

Let us introduce the moments for f:

$$(6) \qquad M_{\mathbf{k}}(f) := \int_{\mathcal{T}_{N-1}} \mathbf{p}^{\mathbf{k}} f(\mathbf{p})\, d\mathbf{p} = \int_{\mathcal{T}_{N-1}} p_1^{k_1}\, p_2^{k_2} \, \dots \, p_{N-1}^{k_{N-1}}\, f(\mathbf{p})\, d\mathbf{p},$$

with $\mathbf{k} := (k_1, k_2, \dots, k_{N-1})$.

In the final form of the equation (4), the only integral terms are the first moments $M_{\mathbf{e}_i}$:

$$(7) \qquad \partial_t f(t, \mathbf{p}) = f(t, \mathbf{p}) \left(\sum_{i=1}^{N-1} (p_i - M_{\mathbf{e}_i}(f)) \left(\upsilon_i + \sum_{j=1}^{N-1} \vartheta_{i,j}\, M_{\mathbf{e}_j}(f) \right) \right).$$

where

- $\mathbf{e}_i \in \mathbb{R}^{N-1}$ is the standard unit vector
- $\vartheta_{i,j} := a_{i,j} - a_{i,N} - a_{N,j} + a_{N,N}$
- $\upsilon_i := a_{i,N} - a_{N,N}$.

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A continuous mixed strategies model

We consider the Cauchy problem

$$(8) \quad \begin{cases} \partial_t f(t,\mathbf{p}) = f(t,\mathbf{p}) \left(\sum_{i=1}^{N-1} (p_i - M_{\mathbf{e_i}}(f)) \left(\upsilon_i + \sum_{j=1}^{N-1} \vartheta_{i,j} M_{\mathbf{e_j}}(f) \right) \right) \\ f(0,\mathbf{p}) = f_0(\mathbf{p}), \end{cases}$$

with $f_0(\mathbf{p}) \geq 0$ and $\int_{\mathcal{T}_{N-1}} f_0(\mathbf{p}) d\mathbf{p} = 1$.

We obtained

- **Global existence of solutions**: using the local existence (based on the Banach-Caccioppoli theorem) and a priori estimate of solutions
- **Asymptotic behaviour of solutions** for two pure strategies games
- **Stationary solutions** for two and three pure strategies games
- **Numerical simulations** for two and three pure strategies games.

A continuous mixed strategies model: global existence of solution

Proposition (Local existence)

For all $M > 0$ *there exists* $T(M) > 0$ *such that if* $||f_0(\mathbf{p})|| \le M$*, then there exists a unique solution* $f \in C([0, \tilde{T}] \times \mathcal{T}_{N-1})$*, for all* $\tilde{T} \le T(M)$*, for the Cauchy problem (8)*

Lemma

The solution f *of the Cauchy problem (8) verifies the a priori estimate* $||f(t, \mathbf{p})||_{L^\infty} \le \max_{\mathbf{p}}(f_0(\mathbf{p}))\, e^{\mathcal{B}t}$*, with* $\mathcal{B} := \sum_{i=1}^{N-1} \left(|\upsilon_i| + \sum_{j=1}^{N-1} |\vartheta_{i,j}| \right)$.

Theorem (Global existence)

There exists a unique global solution $f \in C([0, +\infty) \times \mathcal{T}_{N-1})$ *to Cauchy problem (8).*

A continuous mixed strategies model: global existence of solution

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A continuous mixed strategies model: asymptotic behavior of the solutions for 2×2 games

We consider a game with 2 pure strategies and the payoff matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

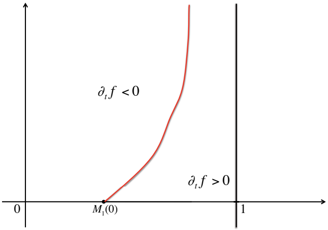
The related Cauchy problem (8) is

$$(9) \qquad \begin{cases} \partial_t f(p) = f(p) \left[(\alpha M_1(f) + \beta)(p - M_1(f)) \right] & t \geq 0,\ p \in \mathcal{T}_1, \\ f(0,p) = f_0(p), & p \in [0,1], \end{cases}$$

with $\alpha = a + d - b - c$, $\beta = b - d$ and $\int_0^1 f_0(p)dp = 1$. The simplex $\mathcal{T}_1$ is just the interval $[0,1]$ and so we have a population where individuals are going to play the first pure strategy with probability $p \in [0,1]$ and the second pure strategy with probability $1 - p$.

A continuous mixed strategies model: asymptotic behavior of the solutions for 2×2 games

(α, β) such that $\beta > \min(0, -\alpha)$



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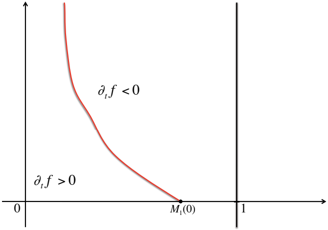
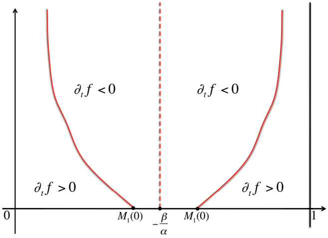


(α, β) such that $\alpha > 0$, $-\alpha < \beta < 0$



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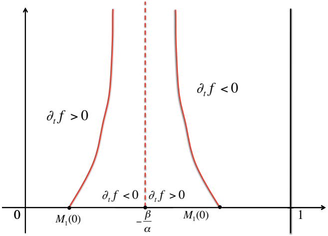

A continuous mixed strategies model: stationary solutions for 2×2 games

- If $-\frac{\beta}{\alpha} \in (0,1)$ then a stationary solution is given by every density function $\overline{f}(p)$ such that $M_1(\overline{f}) = -\frac{\beta}{\alpha}$.
  These stationary solutions are not linearly stable.
- For more general solutions we can say that another class of stationary solutions is given by concentrated Dirac masses, i.e. $f_{\overline{p}} = \delta(p = \overline{p})$. For general perturbations we have three cases
  - Dirac mass concentrated in $\overline{p} = 0$ is stable if $\beta < 0$ $(b < d)$
  - Dirac mass concentrated in $\overline{p} = 1$ is stable if $\alpha + \beta > 0$ $(a > c)$
  - Dirac mass concentrated in $\overline{p} = -\frac{\beta}{\alpha} \in (0,1)$ is stable.

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A continuous mixed strategies model: numerical simulations

For the numerical simulations we consider the PD game with the payoff matrix

$$A = \begin{pmatrix} 1 & 0 \\ b & \varepsilon \end{pmatrix},$$

with $b = 1.1$ and $\varepsilon = 0.001$.

In this case we have $\alpha = 1 - b + \varepsilon < 0$ and $\beta = -\varepsilon < 0$. This means that stationary solutions are expected to be given by Dirac mass concentrated in $\overline{p} = 0$ that is linearly stable.

For numerical simulations we used a trapezoidal rule for the integral term and a fourth order Runge-Kutta method for time discretization, with constant time stepping on the interval $[0, T]$.

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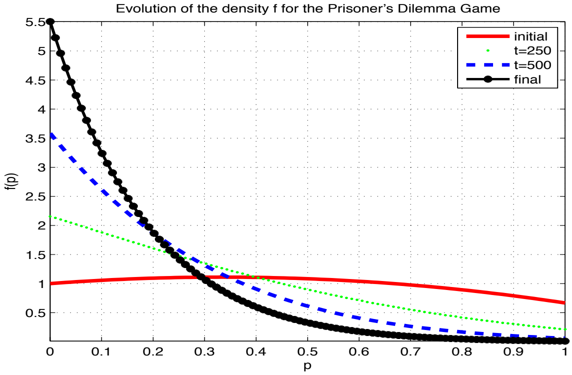
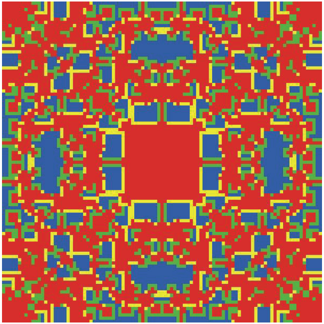


Figure: Prisoner's Dilemma Game: $b = 1.1$, $\varepsilon = 0.001$. Evolution over time of $f(t, p)$ for the equation (7) with initial datum $f_0(p) = -p^2 + \frac{2}{3}p + 1 \quad \forall\, p \in [0, 1]$, for $T = 1000$.

A discrete cellular automata



The discrete approach of cellular automata is based on modelling of the networks of interacting players. In the simplest case, players are assumed to be located at each vertices of a given lattice. At each of the (discrete) time step, each player engages in pairwise interactions with each of co-players from some neighbourhood. Then, players update their strategies according to some rules.

A reaction-diffusion model

Vickers proposed the following reaction-diffusion equation for a game with r pure strategies

$$(10) \qquad \frac{\partial n_i}{\partial t} = n_i \left[\frac{e_i \cdot A\mathbf{n}}{N} - \frac{\mathbf{n} \cdot A\mathbf{n}}{N^2} \right] + D_i \triangle n_i, \qquad i = 1, \ldots, r$$

where

- $n_i(\mathbf{x}, t)$ is the population density for each strategy
- $\mathbf{n} = (n_1, \ldots, n_r)^T$ is the vector of the partial densities
- $N(\mathbf{x}, t) = \sum n_i$ is the total density
- $A \in \mathbb{R}^{r \times r}$ is the payoff matrix
- the ith strategy is denoted by the vector e_i that has 1 in the ith component and zeros elsewhere
- D_i is the diffusion coefficient.

Joint work with A.L.Amadori and R.Natalini [ABN12].
We consider a 2×2 games and denote by n_1 and n_2, respectively, the density of population adopting the first and the second strategy. We adopt an hyperbolic model to describe the dynamics of the partial densities:

$$\partial_t n_i + \partial_x w_i = n_i \left(\frac{\mathbf{e_i} \cdot A\mathbf{n}}{N} - \frac{\mathbf{n} \cdot A\mathbf{n}}{N^2} \right) \qquad i = 1, 2, \tag{11}$$

with

- $N = n_1 + n_2$ the total density of the population.
- $\mathbf{n} = (n_1, n_2)^T$ the vector of partial densities.
- $\mathbf{w} = (w_1, w_2)^T$ is the flux vector of the partial densities.

We need an equation for the fluxes w_i, $i = 1, 2$.

A hyperbolic model for spatial evolutionary games

The Fick's law for the flux is

$$\lambda_i^2 \partial_x n_i = -w_i, \qquad i = 1, 2, \tag{12}$$

To force a finite speed of propagation, we consider a relaxation time τ

$$w_i(t + \tau, x) = -\lambda_i^2 \partial_x n_i, \tag{13}$$

and so we obtain a flux of Cattaneo type, namely

$$\tau \partial_t w_i + \lambda_i^2 \partial_x n_i = -w_i, \qquad i = 1, 2. \tag{14}$$

Finally, the model reads as

$$\begin{cases} \partial_t n_i + \partial_x w_i = n_i \left(\dfrac{\mathbf{e_i} \cdot A\mathbf{n}}{N} - \dfrac{\mathbf{n} \cdot A\mathbf{n}}{N^2} \right), & i = 1, 2, \\ \tau \partial_t w_i + \lambda_i^2 \partial_x n_i = -w_i, & i = 1, 2. \end{cases} \tag{15}$$

A hyperbolic model for spatial evolutionary games: analytical properties

The model (15) is symmetrizable (and strictly hyperbolic): the Riemann invariants are $R_i^{\pm} = n_i \pm w_i \sqrt{\tau}/\lambda_i$ for $i = 1, 2$. The original variables can be recovered from the Riemann invariants as

$$n_i = \frac{1}{2}(R_i^+ + R_i^-), \qquad w_i = \frac{\lambda_i}{2\sqrt{\tau}}(R_i^+ - R_i^-).$$

W.r.t. the Riemann invariants, model (15) has the diagonal form:

$$\partial_t \mathbf{R} + \Theta\, \partial_x \mathbf{R} = \mathbf{F}(\mathbf{R}), \tag{16}$$

where

- $\mathbf{R} := (R_1^+, R_1^-, R_2^+, R_2^-)^T \in \mathbb{R} \times [0, +\infty)$,
- $\Theta := diag(\theta_1^+, \theta_1^-, \theta_2^+, \theta_2^-)$ with $\theta_i^{\pm} := \pm\lambda_i/\tau$
- $\mathbf{F} := (f_1^+, f_1^-, f_2^+, f_2^-)^T \in \mathbb{R}^4$ with

$$\begin{aligned} f_i^{\pm}(\mathbf{R}) = \quad & (-1)^i \frac{(R_1^+ + R_1^-)(R_2^+ + R_2^-)\left(\alpha_1(R_1^+ + R_1^-) + \alpha_2(R_2^+ + R_2^-)\right)}{2(R_1^+ + R_1^- + R_2^+ + R_2^-)^2} \\ & \mp \frac{1}{2\tau}(R_i^+ - R_i^-). \end{aligned}$$

A hyperbolic model for spatial evolutionary games: analytical properties

Using the diagonal model related to the Riemann invariants, we obtain

- **Global existence of broad solutions**: using the standard theory related to a globally Lipschitz continuous source term and a priori estimate of the L^∞ norm of solutions
- **A sequence that approximates solutions**: using a splitting method based on the classical Trotter product formula
- **Long-time behaviour of solutions**.

A nonlinear hyperbolic Trotter formula

We partition the time interval into small steps of type $[k/n, (k+1)/n)$, as $k \in \mathbb{N}$. At the beginning of each time step, we update the initial datum by means of

$$\mathbf{R}_\mathbf{n}^\mathbf{k}(x) := \left(\mathcal{S}_{\frac{1}{n}} \mathcal{F}_{\frac{1}{n}}\right)^k \mathbf{R}_{\mathbf{0n}}(x), \quad \mathbf{R}_{\mathbf{0n}} \to \mathbf{R}_\mathbf{0} \text{ in } L^1(\mathbb{R}; \mathbb{R}^4) \text{ and pointwise a.e.,}$$

where $\mathcal{S}_t$ is the semigroup associated to the Cauchy problem for the diagonal linear hyperbolic system

$$(17) \qquad \begin{cases} \partial_t \mathbf{R} + \Theta\, \partial_x \mathbf{R} = 0, & x \in \mathbb{R}, \quad t > 0, \\ \mathbf{R}(x, 0) = \mathbf{R}_\mathbf{0}(x), & x \in \mathbb{R}, \end{cases}$$

and $\mathcal{F}_t$ is the flow associated to Cauchy problem for the O.D.E. Inside the interval $(k/n, (k+1)/n)$, we follow the evolution operator by setting

$$(18) \qquad \mathbf{R}_\mathbf{n}(x, t) := (\mathcal{S}_{\sigma_n} \mathcal{F}_{\sigma_n})\, \mathbf{R}_\mathbf{n}^\mathbf{k}(x) \qquad \text{for } \sigma_n(t) = t - k/n \in [0, 1/n)\,.$$

that converges to the broad solution of the problem.

A hyperbolic model for spatial evolutionary games: analytical properties

Theorem (Global estimates)

Take $\mathbf{R_0} \in L^1(\mathbb{R};\mathbb{R}^4) \cap (L^\infty(\mathbb{R};\mathbb{R}))^4 \cap BV(\mathbb{R};\mathbb{R}^4)$ *such that* $\mathbf{R_0}(x) \in [0, M_1]^2 \times [0, M_2]^2$, *for a.a.* $x \in \mathbb{R}$, *and let* $\mathbf{R}(t)$ *be the global broad solution to the model. We denote* $\alpha_i = a_{2i} - a_{1i}$ *as* $i = 1, 2$.

- *If* $\alpha_1, \alpha_2 > 0$, *then*
  $0 \leq R_1^\pm(x,t) \leq M_1 \quad 0 \leq R_2^\pm(x,t) \leq M_2 + M_1\overline{\alpha}t$ *for a.a.* $x \in \mathbb{R}\, \forall t \geq 0$.
- *If* $\alpha_2 < 0 < \alpha_1$, *then* $0 \leq R_1^\pm(x,t) \leq \max\left\{M_1, \left|\frac{\alpha_2}{\alpha_1}\right| M_2\right\}$,
  $0 \leq R_2^\pm(x,t) \leq \max\left\{M_2, \left|\frac{\alpha_1}{\alpha_2}\right| M_1\right\}$ *for a.a.* $x \in \mathbb{R}, \forall t \geq 0$.
- *If* $\alpha_1 < 0 < \alpha_2$, *then*
  $0 \leq R_1^\pm(x,t) \leq M_1 e^{\frac{\alpha_1^2}{|\alpha_2|}t} \quad 0 \leq R_2^\pm(x,t) \leq M_2 e^{\frac{\alpha_2^2}{|\alpha_1|}t}$ *for a.a.* $x \in \mathbb{R} \quad \forall t \geq 0$.

Joint work with A.L. Amadori and R. Natalini [ABN10].
We consider the 1D Neumann boundary value problem related to our hyperbolic model

$$(19)\quad \begin{cases} \partial_t \mathbf{R} + \Theta \partial_x \mathbf{R} = \mathbf{F}(\mathbf{R}) & (x,t) \in [-L, L] \times [0, T], T > 0, \\ \mathbf{R}(x,0) = \mathbf{R_0}(x) & x \in [-L, L] \\ R_i^+(\pm L, t) = R_i^-(\pm L, t) & t \in [0, T], \end{cases}$$

and to the reaction-diffusion model proposed by Vickers et alia

$$(20)\quad \begin{cases} \partial_t n_i - D_i \partial_{xx} n_i = (-1)^i g(\mathbf{n}) & (x,t) \in [-L, L] \times [0, T] \\ n_i(x,0) = n_{0i}(x) & x \in [-L, L], \\ \partial_x n_i(\pm L, t) = 0 & t \in [0, T], \end{cases}$$

with $g(\mathbf{n}) = \dfrac{n_1 n_2}{(n_1 + n_2)^2} (\alpha_1 n_1 + \alpha_2 n_2)$ and $\alpha_i = a_{2i} - a_{1i}$ as $i = 1, 2$.

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1D models for spatial evolutionary games: numerical results

In order to perform numerical simulations, we consider the PD game. We discretize the models using the Upwind scheme and the Upwinding of the source term for the hyperbolic model, an the 1D Crank Nicholson scheme for the reaction-diffusion model. The initial data for the two population densities are functions with compact support in $[-\pi, \pi]$:

$$n_{0,1}(x) = \begin{cases} 1 & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ |sin(\pi x)| & \{-1 \leq x \leq -\frac{1}{2}\} \cup \{\frac{1}{2} \leq x \leq 1\} \\ 0 & \text{elsewhere} \end{cases} \tag{21}$$

$$n_{0,2}(x) = \begin{cases} -40x^2 + 164x - 168 & 2 \leq x \leq 2.1 \\ 0 & \text{elsewhere.} \end{cases} \tag{22}$$

1D models for spatial evolutionary games: numerical results

The hyperbolic model

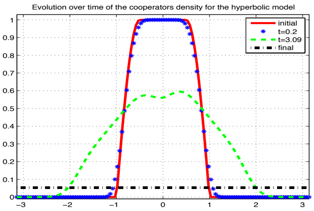

Figure: Cooperators density in the spatial interval $[-\pi, \pi]$ for the initial boundary value problem (19) with $\tau = 0.8$, $L = \pi$, $\lambda_1 = 0.3$, $\lambda_2 = 1$ and $T = 100$.

The reaction-diffusion model

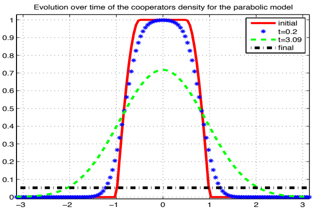

Figure: Cooperators density in the spatial interval $[-\pi, \pi]$ for the initial boundary value problem (20) with $L = \pi$, $D_1 = (\lambda_1)^2 = (0.3)^2$, $D_2 = (\lambda_2)^2 = 1$ and $T = 100$.

The hyperbolic model

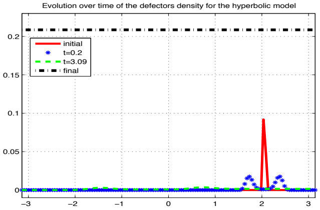

Figure: Defectors density in the spatial interval $[-\pi, \pi]$ for the initial boundary value problem (19) with $\tau = 0.8$, $L = \pi$, $\lambda_1 = 0.3$, $\lambda_2 = 1$ and $T = 100$.

The reaction-diffusion model

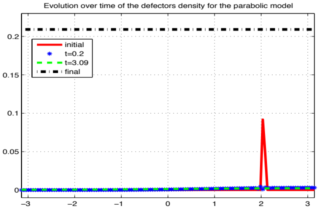

Figure: Defectors density in the spatial interval $[-\pi, \pi]$ for the initial boundary value problem (20) with $L = \pi$, $D_1 = (\lambda_1)^2 = (0.3)^2$, $D_2 = (\lambda_2)^2 = 1$ and $T = 100$.

The L^1 mass of cooperators

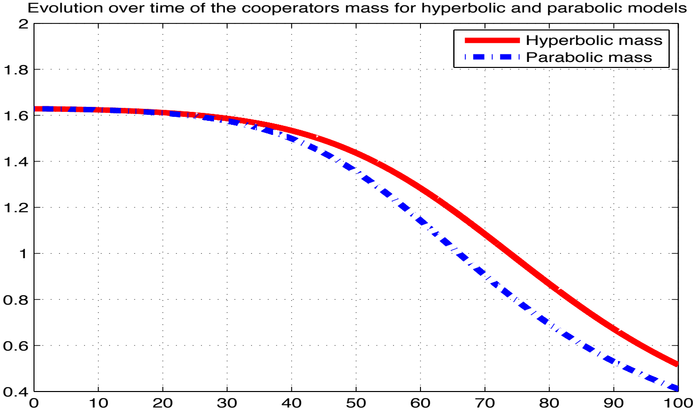

Figure: We plot the evolution over time of the L^1 cooperators mass for the initial hyperbolic and parabolic boundary value problems.

2D models for spatial evolutionary games: numerical results

We consider a 2D hyperbolic model with Neumann boundary conditions

$$(23)\quad \begin{cases} \partial_t n_i + \nabla \cdot w_i = (-1)^i g(\mathbf{n}) & \mathbf{x} \in \Omega := [0,1] \times [0,1],\ t \in [0,T] \\ \tau \partial_t w_i + \lambda_i^2 \nabla n_i = -w_i, & i = 1,2, \\ \nabla n_i \cdot \overrightarrow{n}|_{\partial\Omega} = 0, \\ w_i \cdot \overrightarrow{n}|_{\partial\Omega} = 0, \end{cases}$$

and the 2D reaction-diffusion model proposed by Vickers et alia

$$(24)\quad \begin{cases} \partial_t n_i - D_i \partial_{xx} n_i = (-1)^i g(\mathbf{n}), & \mathbf{x} \in \Omega,\ t \in [0,T] \\ \nabla n_i \cdot \overrightarrow{n}|_{\partial\Omega} = 0, & i = 1,2, \end{cases}$$

with $n_i : \mathbb{R}_+^2 \times \mathbb{R}_+ \to \mathbb{R}$ and $w_i : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$.

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and the 2D reaction-diffusion model proposed by Vickers et alia

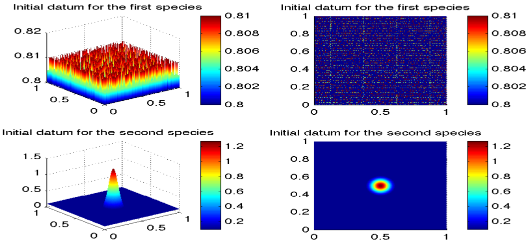
$$(24)\quad \begin{cases} \partial_t n_i - D_i \partial_{xx} n_i = (-1)^i g(\mathbf{n}), & \mathbf{x} \in \Omega,\ t \in [0,T] \\ \nabla n_i \cdot \overrightarrow{n}|_{\partial\Omega} = 0, & i = 1,2, \end{cases}$$

with $n_i : \mathbb{R}_+^2 \times \mathbb{R}_+ \to \mathbb{R}$ and $w_i : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$.

2D models: numerical results for PD game

For numerical simulations we consider the PD game and discretize the models using a Relaxation method with 5 velocities for the hyperbolic model and the Crank Nicholson scheme for the reaction-diffusion model. The initial data for densities are

$$
\begin{aligned}
n_{1,0}(\mathbf{x}) &\in [0.8, 0.81] \quad \forall \mathbf{x} \in \Omega, \\
n_{2,0}(\mathbf{x}) &= max\{0.1,\ 8\left(\frac{1}{2\pi} e^{-190[(x_1-\frac{1}{2})^2+(x_2-\frac{1}{2})^2]}\right)\} \quad \forall \mathbf{x} = (x_1, x_2) \in \Omega.
\end{aligned}
$$



The reaction-diffusion model

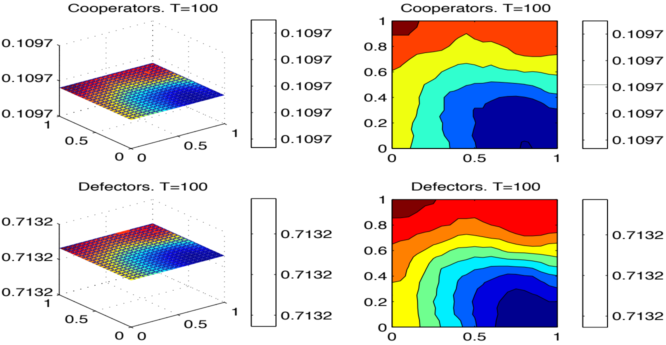


Figure: Prisoner's Dilemma Game: we plot the final configuration for the cooperators density (figures above) and the defectors densities (figures below), in the spatial domain $\Omega = [0,1] \times [0,1]$, related to the reaction-diffusion model (24). The parameters are $D_1 = D_2 = 0.25$ and $T = 100$.

The hyperbolic model

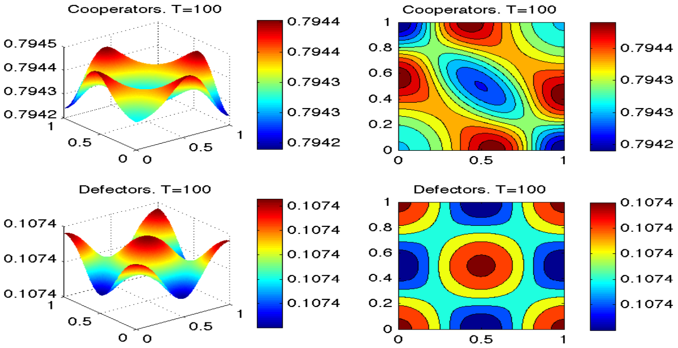


Figure: Prisoner's Dilemma Game: we plot the configuration for the cooperators density and the defectors density at time $T = 50$ and $T = 100$, in the spatial domain $\Omega = [0,1] \times [0,1]$, related to the hyperbolic model (23). The parameters are $\tau = 0.8$, $\lambda_1 = \lambda_2 = 0.5$ and $T = 100$.

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