A SYMMETRY RESULT IN A FREE BOUNDARY PROBLEM

C. Trombetti – University of Naples "Federico II"

Mostly maximum principle

4th edition

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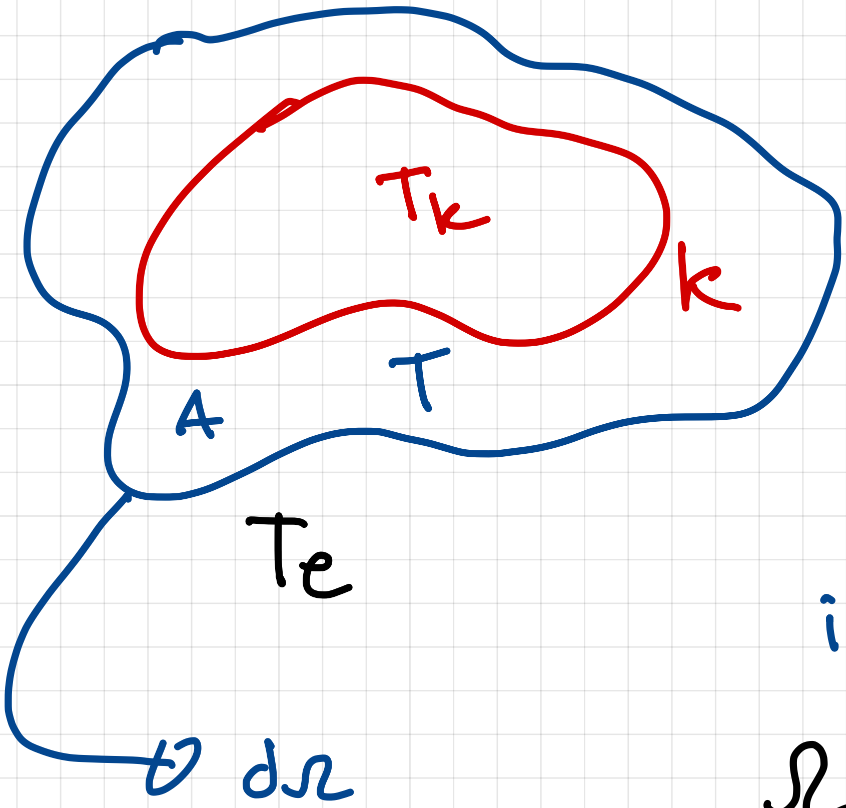
JOINT WORK WITH: D. BUCUR – M. NAHON – C. NITSCH
(CALC. VAR. TO APPEAR)

$K \subseteq \mathbb{R}^n$ $\quad K \subseteq \Omega$ $\quad \Omega$ open; Lipschitz; $\beta > 0$

(P1) $$E(K,\Omega) = \min \left\{ \int_\Omega |\nabla v|^2 + \beta \int_{\partial\Omega} v^2 \quad \begin{array}{l} v \in H^1(\Omega) \\ v = 1 \text{ on } K \end{array} \right\}$$

If u is a minimizer then u satisfies

(Eq.1) $$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus K \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \partial\Omega \cap \{u > 0\} \setminus \partial K \\ u = 1 & \text{on } K \end{cases}$$



body with fixed temperature T_k

surrounded by an insulator A. T is the temperature inside A

$$\Omega = k \cup A$$

Assume $T_k > T_e$

$$u = \frac{T - T_e}{T_k - T_e}$$

We are interested in the following optimization problem:

(P2)
$$\inf_{\substack{K \subseteq \Omega \\ |K| = \omega_n \\ |\Omega| \leq \omega_n R^n = M}} E_\beta(K, \Omega)$$

$R \geq 1$; ω_n = volume of the unit ball.

Theorem 1 The solution to (P2) exists and consists of two concentric balls. The radius of the outer ball equals R or 1 according to $\min\{E_\beta(B_1, B_1) ; E_\beta(B_1, B_R)\}$. Moreover the associated state function u is radially symmetric.

THE CONVECTION CASE.

(P1) $$E_\beta(K,\Omega) = \inf_{\substack{v\in H^1(\Omega)\\ v\geq 1_K}} \int_\Omega |\nabla v|^2 + \beta \int_{\partial\Omega} v^2$$

FIRST STEP: STUDY OF THE RADIAL CASE

$$R \in [1,+\infty[\longrightarrow E_\beta(B_1, B_R)$$

- $$\phi_n(\rho) = \begin{cases} \log \rho & \text{if } n=2 \\ -\dfrac{1}{(n-2)\rho^{n-2}} & \text{if } n\geq 3 \end{cases}$$

(ϕ_n is increasing)

The solution u^* to (P1), taking into account the boundary conditions, is the following

$$u(x) = 1 - \frac{\beta\left(\phi_n(x) - \phi_n(1)\right)_+}{\phi'_n(R) + \beta\left(\phi_n(R) - \phi_n(1)\right)}$$

and the associated energy is

$$E_\beta(B_1, B_R) = \frac{\beta \cdot \mathrm{Per}(B_1)\, \phi'_n(1)}{\phi'_n(R) + \beta\left(\phi_n(R) - \phi_n(1)\right)}$$

$$= \frac{\beta\, n\, \omega_n}{\phi'_n(R) + \beta\left[\phi_n(R) - \phi_n(1)\right]}$$

In particular :

- $\frac{d}{dR} E(B_1, B_R) \leq 0 \Leftrightarrow \frac{d}{dR}\left(\phi_n'(R) + \beta\,\phi_n(R)\right) \geq 0$

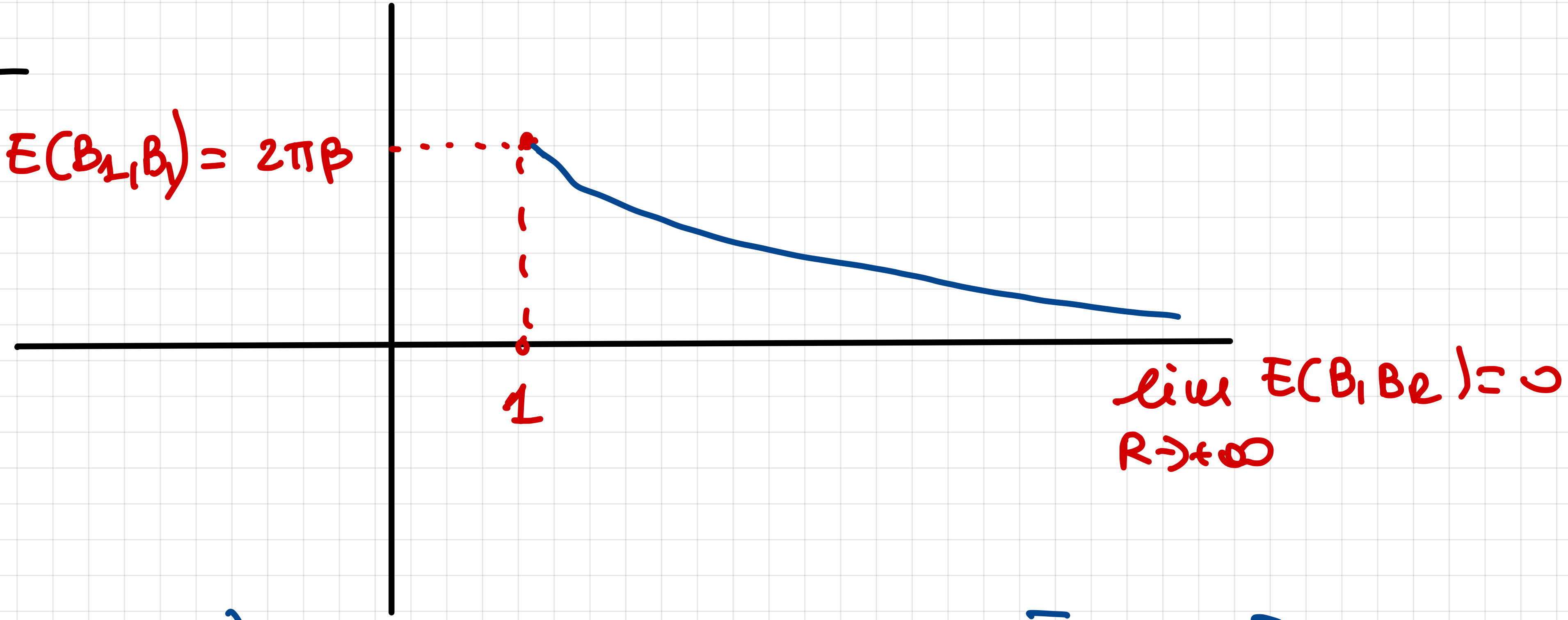
$\Leftrightarrow \quad R \geq \frac{n-1}{\beta}$

- - $E_\beta(B_1, B_1) = \boxed{\beta\, n\, \omega_n}$; $\lim_{R \to +\infty} E(B_1, B_R) = \boxed{n(n-1)\omega_n}$

$\Downarrow$

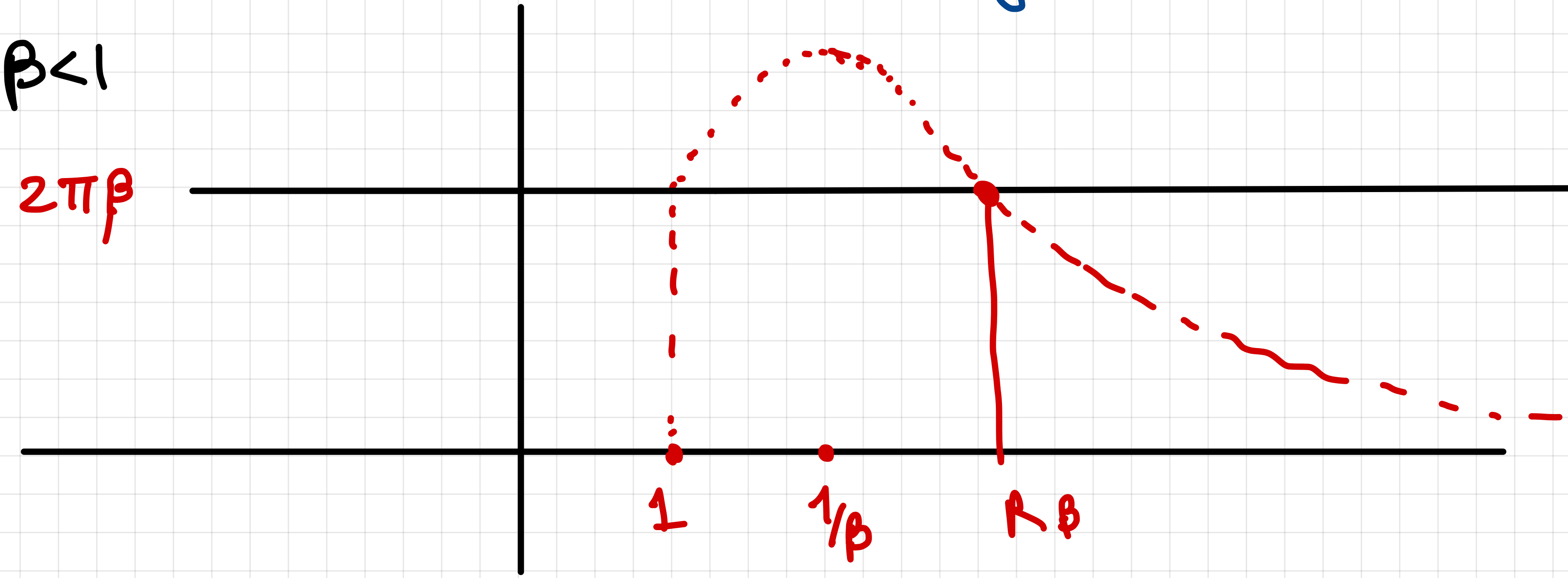
In dimension $n = 2$ two cases occur

$\beta \geq 1$



$E(B_1, B_R)$ is decreasing in $[1, +\infty[$

$0 < \beta < 1$

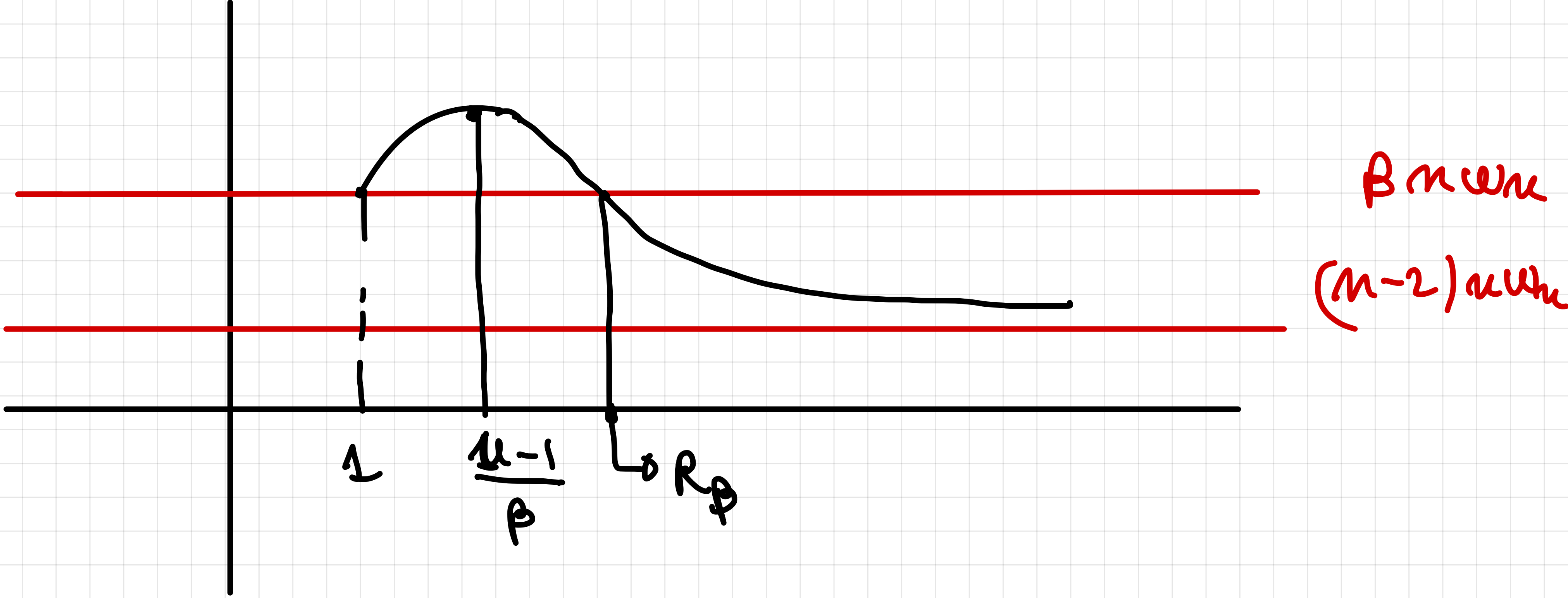


In this case $E(B_1, B_R)$ increases in the interval $[0, 1/\beta]$ and decreases in the interval $[1/\beta, +\infty[$. Moreover there exists a unique $R_\beta > 1/\beta$: $E(B_1, B_{R_\beta}) = E_\beta(B_1, B_L)$

In Dimension $n \geq 3$ Three cases can occur:

- $\beta \geq n-1 \Rightarrow E(B_1, B_R)$ decreases on $[1, +\infty[$

$n-1 < \beta < n-1$



$E(B_1, B_R)$ is increasing on $[1, \frac{n-1}{\beta}]$, decreasing on $[\frac{n-1}{\beta}, +\infty[$ with the existence of a unique $R_\beta > \frac{n-1}{\beta}$: $E(B_1, B_{R_\beta}) = E(B_1, B_1)$

if $\beta \leq n-2$ Then $E_\beta(B_1, B_R)$ reaches its minimum at $R=1$

Lemma: Let v^* be the minimizer to

$$E_\beta(B_1, B_R) = \inf_{\substack{v \in H^1(B_R) \\ v \geq 1_{B_1}}} \int_{B_R} |\nabla v|^2 + \beta \int_{B_R} v^2$$

$$\Rightarrow \quad \boxed{\frac{|\nabla u^*|}{u^*} \leq \beta} \text{ on } B_R \setminus B_1 \text{ iff}$$

$$\boxed{E_\rho(B_1, B_\rho) \geq E_\beta(B_1, B_R)} \quad \forall \rho \in [1, R]$$

SKETCH OF THE PROOF of Th 1.

Let $K \subseteq \Omega$ $|K| = \omega_n$ $|\Omega| \leq M = \omega_n R^n$

Let u be the minimizer to $E_\beta(K,\Omega)$.

Assume K and Ω smooth. Denote by

$$\Omega_t = \{x \in \Omega : u(x) > t\}$$

$$\partial\Omega_t = \partial^i\Omega_t \cup \partial^e\Omega_t = \left(\{u=t\} \cap \Omega\right) \cup \left(\partial\Omega_t \cap \partial\Omega\right)$$

for a.e. $t \in (0,1)$

$$0 = \int_{t<u<1} \frac{\Delta u}{u}\,dx = \int_{t<u<1} \left[\operatorname{div}\left(\frac{\nabla u}{u}\right) - \nabla u \cdot \nabla \frac{1}{u}\right] dx$$

$$= \int_{\partial(t<u<1)} \frac{\nabla u}{u} \cdot \nu_{\Omega_t} \, d\mathcal{H}^{n-1} + \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} \, dx =$$

$$\frac{\nabla u}{u} \cdot \nu = -\beta \text{ on } \partial\Omega$$

$$= \int_{\partial K \cap \Omega} |\nabla u| \, d\mathcal{H}^{n-1} - \int_{\partial^i \Omega_t} \frac{|\nabla u|}{u} \, d\mathcal{H}^{n-1} - \int_{\partial\{t<u<1\} \cap \partial\Omega} \beta \, d\mathcal{H}^{n-1}$$

$$+ \int_{\Omega_t} \frac{|\nabla u|^2}{u^2}$$

$$\Rightarrow \int_{\partial K \cap \Omega} |\nabla u| \, d\mathcal{H}^{n-1} = \int_{\partial^i \Omega_t} \frac{|\nabla u|}{u} \, d\mathcal{H}^{n-1} +$$

$$\beta \mathcal{H}^{n-1}(\partial\{t<u<1\} \cap \partial\Omega) - \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} \, dx$$

On the other hand

$$E_\beta(K,\Omega) = \int_\Omega |\nabla u|^2 + \beta \underset{\partial\Omega}{\int} u^2 =$$

$$-\beta u^2$$

$$= \underbrace{\int_{\partial\Omega \setminus \partial K} u \cdot \frac{\partial u}{\partial \nu}} + \int_{\partial K \cap \Omega} |\nabla u| + \beta \underbrace{\int_{\partial\Omega} u^2}$$

$$= \int_{\partial K \cap \Omega} |\nabla u| + \beta \, \mathcal{H}^{n-1}(\partial K \cap \partial \Omega)$$

$$\Rightarrow \quad E_\beta(K,\Omega) = \beta \, \mathcal{H}^{n-1}(\partial^e \Omega_t) + \int_{\partial^i \Omega_t} \frac{|\nabla u|}{u} - \int_{\Omega_t} \frac{|\nabla u|^2}{u^2}$$

for a.e. $t \in (0,1)$

Denote by $H(t,\phi) = \beta \mathcal{H}^{n-1}(\partial^e \Omega_t) + \int_{\partial^i \Omega_t} \phi \, d\mathcal{H}^{n-1} - \int_{\Omega_t} \phi^2 \, dx$

Lemma 2

$\forall \phi \in C^\infty \; ; \; \phi \geq 0 \quad \Rightarrow \quad \exists \; t \in]0,1[\; :$

$$H(t,\phi) \leq E_\beta(K,\Omega).$$

Obs

$$E_\beta(K,\Omega) = H\left(t, \frac{|\nabla u|}{u}\right) \quad \forall \text{ a.e. } t \in (0,1)$$

Let u^* be the solution to $(P1)$ in $(B_1, B_R) \Rightarrow$

$$E_\beta(B_1, B_R) = H\left(t, \frac{|\nabla u^*|}{u^*}\right)$$

assume $\dfrac{|\nabla u^*|}{u^*} \leq \beta$ in $B_R \setminus B_1$

deed let $\phi^*(x) = \dfrac{|\nabla u^*|}{u^*} = g(|x|)$

For $t \in (0,1)$ let $B_{r(t)}$: $|B_{r(t)}| = |\Omega_t|$. If $x \in \Omega \setminus K$

$\phi(x) = g(r(t)) \cdot \leq \beta$ $\qquad \phi_{\partial\Omega^t} = \phi^*_{\partial B_{r(t)}}$

- $H^{n-1}(\partial B_{r(t)}) \leq H^{n-1}(\partial \Omega_t)$ $\qquad \therefore \displaystyle\int_{\Omega_t} \phi^2 = \int_{B_{r(t)}} \frac{|\nabla u|^{*2}}{u^{*2}}$

Let t: $\quad H(t,\phi) \leq E_\beta(K,\Omega)$

$\quad\; \shortparallel$

$E_\beta(K,\Omega) \geq \beta H^{n-1}(\partial^e \Omega_t) + \displaystyle\int_{\partial^* \Omega_t} \phi - \int_{\Omega_t} \phi^2 \geq$

$\geq \displaystyle\int_{\partial\Omega^t} \phi - \int_{B_{r(t)}} \phi^{*2}(x)\,dx = \int_{\partial\Omega^t} \phi^*_{\partial B_{r(t)}} - \int_{B_{r(t)}} \phi^{*2}$

isop. inep.

$$\geq \int_{\partial B_2(t)} \phi^*_{\partial B_2(t)} - \int_{B_2(t)} \phi^{*2} =$$

$$= H\left(u^*\Big|_{\partial B_2(t)}, \frac{|\nabla u^*|}{u^*}\right) = E_\beta(B_1, B_R)$$

main assumption : $\frac{|\nabla u^*|}{u^*} \leq \beta$

- in the case $\beta \geq n-1$ $E(B_1, B_\rho) \geq E(B_1, B_R)$ $\forall \rho \leq R$ since $\rho \to E(B_1, B_\rho)$ is decreasing

•• In the case $n-2 < \beta < n-1$ if $\boxed{R \geq R_\beta}$

$E(B_1, B_p) \geq E(B_1, B_R)$

... If $R \leq R_\beta$ u^* is the one associated to (B_1, B_{R_β}). Then

$$E(B_1, B_p) \geq E(B_1, B_{R_\beta}) \qquad \forall p \leq R_\beta$$

$$\Rightarrow \quad E(K, \Omega) \geq E(B_1, R_\beta) = E(B_1, B_1)$$

$n \geq 3$ $\quad \beta \leq n-2$ $\quad$ The minimum is $E(B_1, B_1)$

Conclusion :

If $M = \omega_n R^n$ The minimizer is (B_1, B_2):

- if $\beta \geq n-1 \Rightarrow r = R$

- $n-2 < \beta < n-1$ if R_β is the unique solution $E(B_1, B_{R_\beta}) = \beta n \omega_n = E(B_1, B_1)$

if $R > R_\beta \Rightarrow r = R$

$1 \leq R < R_\beta \Rightarrow r = 1$

$R = R_\beta \Rightarrow r = 1$ or $r = R_\beta = R$

THANK YOU FOR YOUR

ATTENTION