Elliptic estimates with optimized constants and two applications to the qualitative theory of elliptic PDE

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The Setting

Let us have a real-valued uniformly elliptic second order operator, either in divergence form

$$\mathcal{L}_D[u] := \mathrm{div}(A(x)Du + b_1(x)u) + b_2(x)\cdot Du + c(x)u,$$

or in non-divergence form

$$\mathcal{L}_{ND}[u] := \mathrm{tr}(A(x)D^2u) + b_1(x)\cdot Du + c(x)u,$$

or more generally a fully nonlinear Hamilton-Jacobi-Bellman operator (i.e. a supremum or an infimum of $\mathcal{L}_{ND}$s), for instance, an extremal operator of Pucci type

$$F[u] := \mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) \pm b(x)|Du| + c(x)u.$$

Let $\mathcal{L}[u]$ denote any of these. We consider weak (super-) solutions of $\mathcal{L}[u] = f$, i.e. weak Sobolev if $\mathcal{L} = \mathcal{L}_D$, viscosity if $\mathcal{L} = \mathcal{L}_{ND}$ or F.

The Setting

We always assume that $A(x) \in L^\infty(\Omega)$ satisfies

$$\text{there exist } 0 < \lambda \le \Lambda \text{ such that } \lambda I \le A(x) \le \Lambda I, \quad x \in \Omega;$$

$$A \in C(\Omega) \text{ if } \mathcal{L} = \mathcal{L}_{ND}.$$

The lower-order coefficients belong locally to Lebesgue spaces which make possible for weak solutions to satisfy the generalized maximum principle and the Harnack inequality; specifically,

$$b, b_1, b_2 \in L^q_{\mathrm{loc}}(\Omega) \text{ for some } q > n,$$

$$c \in L^p_{\mathrm{loc}}(\Omega) \text{ for some } p > p_0, \text{ where}$$

$$p_0 = \begin{cases} n/2, & \text{if } \mathcal{L} = \mathcal{L}_D \\ p_E, & \text{if } \mathcal{L} = \mathcal{L}_{ND} \text{ or } F \end{cases}$$

and $p_E = p_E(n, \lambda, \Lambda) \in (n/2, n)$ is the Escauriaza constant.

The Vázquez Strong Maximum Principle

SMP: A “linear”/“positively homogeneous” result

If u is a nonnegative supersolution and u vanishes at one point, then u vanishes everywhere.

$$\mathcal{L}[u] \leq 0, \quad u \geq 0, \quad u(x_0) = 0 \qquad \text{implies} \qquad u \equiv 0.$$

Or: if a subsolution touches a supersolution from below, then they coincide.

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Vázquez 1984

The zero-order dependence may be non-Lipschitz at zero!

SMP: A “linear”/“homogeneous” result ? No.

Theorem (Vázquez 1984)

The threshold is strictly between u and u^p, $p < 1$:
if $f(0) = 0$, $f \geq 0$ is nondecreasing on $(0, \infty)$, and

$$\int_0 \frac{ds}{\sqrt{F(s)}} = \infty, \qquad F(s) = \int_0^s f(t)\,dt,$$

then for each classical supersolution

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Main example

$$f(s) \sim s\,|\log(s)|^a, \quad a \leq 2.$$

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Remark: The integral condition reminds of the so-called Keller-Osserman condition for existence of entire (sub-)solutions (there the integral is at infinity)

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A series of papers and a book, for very general divergence form operators,

"The maximum principle", Birkhäuser, 2007

by P. Pucci and J. Serrin (also papers in collaboration with H. Zou). They show the integral condition is also necessary, and that it is sufficient that f be nondecreasing only in a right neighborhood of zero. The latter can be removed in some cases (Pucci-Radulescu 2018).

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Felmer-Montenegro-Quaas 2009

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All previous works required operators with *bounded* coefficients, because of the method of proof.

The classical proof of SMP

1. Prove a weak maximum (comparison) principle for $\mathcal{L}$ with $c \leq 0$.
2. Construct a (radial) subsolution in an annulus, with values $0, 1$, and non-vanishing normal derivative on the boundary.
3. Deduce the Hopf lemma, by comparison.
4. Deduce the SMP by contradiction, combining the Hopf lemma with a continuity-connectedness argument.

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The Vázquez integral condition appears in Step 2. In that step we solve an ODE boundary value problem and need radial (constant) coefficients. For instance, to get a subsolution of $\Delta u + b(x)|Du| = f(u)$ we look for a radial solution of $\Delta\phi - \|b\|_\infty |D\phi| = f(\phi)$ with values $0, 1$ on the boundary of the annulus.
The normal derivative of that solution does not vanish precisely under the integral condition on f.

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We prove it holds for nonlinearities as in the main example.

Theorem (P. Souplet, B.S.)

Let u be a nonnegative weak supersolution of

$$\mathcal{L}[u] \leq f(u) \qquad \textit{in } \Omega,$$

where f is continuous on the range of u, $f(0) = 0$, and

$$\limsup_{s \to 0} \frac{f(s)}{s\,(\ln s)^2} < \infty.$$

If $\operatorname{essinf}_B u = 0$ for some ball $B \subset\subset \Omega$ then $u \equiv 0$ in Ω.

An application – problems with natural growth

Existence, behavior and a priori bounds for

$$\begin{cases} \mathcal{L}_0[u] + \langle M(x)\nabla u, \nabla u\rangle + c(x)u & = & h(x) & \text{in} & \Omega \\ u & = & 0 & \text{on} & \partial\Omega \end{cases}$$

Coercive: Kazdan—Kramer, Boccardo—Murat—Puel, Ferone—Murat, dall'Aglio, Giachetti, Puel,Maderna, Pagani, Salsa; Grenon, Porretta; Abdellaoui, Peral; Bidaut-Véron; Boccardo, Gallouet, Murat...

Non-coercive: B.S, L. Jeanjean, D. Arcoya, C. de Coster, K. Tanaka, P. Souplet, A. Fernandez, G. Nornberg, D. Schiera...

Non-divergence/fully nonlinear operators first studied by B.S., B.S. - G. Nornberg (JFA 2018). In particular, it turns out that a crucial a priori bound hinges on the SMP for

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) - b(x)|Du| - c(x)u \leq u|\ln u|.$$

We had to restrict to bounded coefficients because of that. The theorem above extends all our results to the natural and most general framework of operators with integrable coefficients.

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Remark: no need of monotonicity assumption on f.

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If $\operatorname{essinf}_B u = 0$ *for some ball* $B \subset\subset \Omega$ *then* $u \equiv 0$ *in* Ω*.*

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Open problem 1

Does the result hold under Vázquez' integral hypothesis on F ?

At first glance, that hypothesis is quite an ODE one...

(New) Method of proof

The (weak) Harnack inequality

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in which

the constant is optimized
with respect to the domain and the norms of the coefficients of $\mathcal{L}$.

The Weak Harnack inequality

Let $u \geq 0$ be a supersolution of $\mathcal{L}[u] \leq 0$ in B_{R+1}. Then

$$\left(\int_{B_R} u^{\varepsilon} \right)^{1/\varepsilon} \leq C_{WH} \inf_{B_R} u.$$

where ϵ depends on n, λ, Λ,
and C_{WH} depends on $n, p, q, \lambda, \Lambda$, $\underline{\|b\|_q, \|c\|_p, \text{and } R}$.

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De Giorgi, Moser, Trudinger; Krylov-Safonov, Caffarelli, Caffarelli-Cabre, Koike-Swiech...

- Quantifies the SMP;
- Is a ”growth lemma” – implies Hölder regularity of solutions.

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- Quantifies the SMP;
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$$\mathcal{L}[u] \leq f \;\text{ in }\; B_{R+1} \quad \rightarrow \quad \left(\int_{B_R} u^{\varepsilon} \right)^{1/\varepsilon} \leq C_{WH} \left(\inf_{B_R} u + \|f\|_{L^p(B_{R+1})} \right).$$

WHI with an optimized constant

Uniformly local Lebesgue spaces (Kato, Ginibre-Velo):

$$\|h\|_{L^s_{ul}(\Omega)} := \sup_{x\in\mathbb{R}^N} \|h\|_{L^s(\Omega\cap B_1(x))} < \infty$$

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Set, for any r, $n < r \leq \infty$,

$$\beta_r = \frac{1}{1 - (n/r)}, \qquad \gamma_r = \frac{1}{2 - (n/r)}, \qquad \beta_\infty = 1, \quad \gamma_\infty = \frac{1}{2},$$

and

$$M = M_R := 1 + \|b\|^{\beta_q}_{L^q_{ul}(B_{R+1})} + \|c\|^{\gamma_p}_{L^p_{ul}(B_{R+1})}.$$

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$$M = M_R := 1 + \|b\|^{\beta_q}_{L^q_{ul}(B_{R+1})} + \|c\|^{\gamma_p}_{L^p_{ul}(B_{R+1})}.$$

Theorem

$$C_{WH} \leq e^{C_0 MR},$$

for some C_0 depending only on $n, p, q, \lambda, \Lambda$.

Optimality obvious from the ODE $u'' - 2bu' - cu = 0$, $b, c \in \mathbb{R}^+$, $u(x) = e^{Dx}$, $D = b + \sqrt{b^2 + c}$.

$$M := 1 + \|b\|^{\beta q}_{L^q_{ul}(B_{R+1})} + \|c\|^{\gamma p}_{L^p_{ul}(B_{R+1})}.$$

$$\mathcal{L}[u] \le h \ \text{ in } \ B_{R+1} \quad \to \quad \left(\int_{B_R} u^\varepsilon \right)^{1/\varepsilon} \le e^{C_0 MR} \big(\inf_{B_R} u + \|h\|_{L^p_{ul}(B_{R+1})} \big).$$

Idea of proof: Take a ball of size $\sim M^{-1}$, rescale to a ball of size 1. The new operator has coefficients with norms ≤ 1.

Scale back, and cover with overlapping balls, use a Harnack chain optimized with respect to the geometry.

Also

local maximum principle

$$\mathcal{L}[u] \ge h \ \text{ in } \ B_{R+1} \quad \to \quad \sup_{B_R} u \le C_0 \left(M^{n/\varepsilon} \|u\|_{L^\varepsilon_{ul}(B_{R+1})} + M^{(n/p)-2} \|h\|_{L^p_{ul}} \right)$$

Idea of proof of the extension of SMP

$$\mathcal{M}^{\pm}_{\lambda,\Lambda}(D^2u) - b(x)|Du| - (\hat{c}(x) + |\ln(u)|^a)u \le 0,$$

$$c(x) = \hat{c}(x) + |\ln(u(x))|^a.$$

$$A := 1 + \|b\|^{\beta q}_{L^q_{ul}(B_{R+1})} + \|c\|^{\gamma p}_{L^p_{ul}(B_{R+1})}.$$

$$\left(\int_{B_R} u^\varepsilon\right)^{1/\varepsilon} \le e^{C_0 MR} \inf_{B_R} u.$$

Open problem 2

We prove SMP for the "nonlinear" operator $\mathcal{L}[u] - f(u)$ using WHI with an optimized constant for the "linear" equation $\mathcal{L}[u] \leq h(x)$.

Would there be a (W)HI for the 'nonlinear" inequality $\mathcal{L}[u] - f(u) \leq 0$?
Or at least for $\mathcal{L}[u] - f(u) = 0$?

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Or at least for $\mathcal{L}[u] - f(u) = 0$?

Vesa Julin, JMPA 2016: yes, if $\mathcal{L}[u] = \mathrm{div}(A(x)Du) = f(u)$. Then

$$\int_{\inf u}^{\sup u} \frac{ds}{s + \sqrt{F(s)}} \leq C_0.$$

Difficult proof, using the div structure and De Benedetto-Trudinger approach, writing an ODE for the volumes of the super-level sets.

Completely open for other second-order operators (say Pucci), and for operators with lower order coefficients.
Can we prove (something like) this, maybe by a (clever) iteration ?

The Landis conjecture

A question by Kondratiev and Landis (1988):

Is it true that solutions of

$$\Delta u + c(x)u = 0, \qquad c \in L^\infty \qquad \text{in } \mathbb{R}^N \setminus B_1$$

cannot decay super-exponentially at infinity ? Does there exist $M = M(\|c\|_\infty)$, such that if

$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0$$

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STILL OPEN FOR REAL SOLUTIONS!

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Variants:

- What if the solution is defined in the whole $\mathbb{R}^N$? Also still open.
- Kenig's (weaker) conjecture: there are no solutions which decay quicker than $e^{-|x|^{1+\epsilon}}$, $\epsilon > 0$. Also still open for $n > 2$.
- More general operators.

The Landis conjecture

$$\Delta u + c(x)u = 0,$$

$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0 \qquad \Rightarrow \qquad u \equiv 0.$$

Meshkov 1991

If c is complex, then false ! The optimal rate is

$$e^{-|x|^{4/3}}$$

A lot of developments by Bourgain, Kenig, Escauriaza, Ponce, Vega, Davey, Zhu...

Based on (powerful) Carleman type estimates. These do not distinguish between real and complex solutions, so no hope to prove the conjecture in the real case by such techniques.

The Landis conjecture - real case

$$\mathcal{L}[u] = 0,$$
$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0 \qquad \Rightarrow \qquad u \equiv 0.$$

Kenig, Silvestre and Wang 2015: weak form in $\mathbb{R}^2$, for $\mathcal{L}_D$ with bounded coefficients and one of the $b_i = 0$, $c(x) \leq 0$; a quantitative bound, within distance 1 of each point on the sphere $|x| = R$ there is a point at which $|u|$ is at least $\exp(-C_0 R(\log R))$. A bound in $\exp(-C_0 R(\log R)^2)$ for solutions in exterior domains of $\mathbb{R}^2$. This paper brought a number of generalizations, Davey, Kenig, Wang, Zhu...

All these works are for $n = 2$ and equations in divergence form, and make various hypotheses on the lower-order coefficients of $\mathcal{L}_D$ which in particular imply that $\mathcal{L}_D$ or its dual *satisfy the maximum principle on bounded subdomains.*

The Landis conjecture - real case

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$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0 \qquad \Rightarrow \qquad u \equiv 0.$$

L. Rossi 2020: any dimension, $\mathcal{L} = \mathcal{L}_{ND}$, bounded coefficients, $\mathcal{L}_{ND}$ satisfies the maximum principle on bounded subdomains.

Also exterior domains, with a sign condition on the boundary.

Uses comparison, and the fact that $e^{-M|x|}$ is a subsolution for large M – requires bounded coefficients.

Earlier by Arapostathis-Biswas-Ganguly (probability techniques), recent by K. Le Balc'h (duality method of M. Pierre)

The Landis conjecture - real case

$$\Delta u + c(x)u = 0,$$
$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0 \qquad \Rightarrow \qquad u \equiv 0.$$

Logunov-Malinnikova-Nadirashvili-Nazarov 2020, preprint:
Kenig's form of the conjecture is valid in $\mathbb{R}^2$.

The decay is no worse than $e^{-|x|\sqrt{\log(|x|)}}$.

Uses quaseconformal mappings. Higher dimensions still open.

The Landis conjecture - real case

$$\mathcal{L}[u] = 0,$$
$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0 \qquad \Rightarrow \qquad u \equiv 0.$$

Unbounded b_i (with c bounded) – Kenig, Davey, Wang, for divergence form operators and $n = 2$ only,

under the restrictions that b_i are integrable at infinity, i.e. belong to $L^q(\mathbb{R}^2)$, $q > 2$, and that $|u|$ grows at most like $\exp(C_0|x|^\alpha)$ with $\alpha = 1 - 2/q \in (0, 1)$.

These rather strong hypotheses lead to a different Landis type result with stronger conclusion, ruling out solutions that decay like $\exp(-C_1|x|^{\alpha+})$.

The Landis conjecture - real case

$$\mathcal{L}[u] = 0,$$
$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0 \qquad \Rightarrow \qquad u \equiv 0.$$

Our contribution: prove the Landis conjecture in $\mathbb{R}^N$ in any dimension,

for unbounded lower-order coefficients which are only uniformly locally integrable (and thus bounded coefficients are a very particular case),

under the hypothesis that the maximum principle holds in any bounded subdomain.

We also consider exterior domains.

$$\mathcal{L}[u] = 0,$$
$$\lim_{|x|\to\infty} e^{M|x|}|u(x)| = 0 \qquad \Rightarrow \qquad u \equiv 0.$$

Theorem (P. Souplet, B.S.)

Let $\Omega = \mathbb{R}^N$ or Ω be an exterior domain. Assume $\mathcal{L}$ satisfies the maximum principle in each bounded subdomain of Ω. Then there exists a constant $C_0 = C_0(n, p, q, \Lambda/\lambda)$ such that if u is a solution of

$$\mathcal{L}[u] = 0 \text{ in } \Omega, \quad \text{with } u \geq 0 \text{ on } \partial\Omega \text{ or } u \leq 0 \text{ on } \partial\Omega \text{ (if } \partial\Omega \text{ is not empty)}$$

and

$$\lim_{|x|\to\infty} e^{C_1|x|}|u(x)| = 0, \quad \text{with } C_1 := C_0 \left(1 + \|b\|_{L^q_{ul}(\Omega)}^{\frac{1}{1-(n/q)}} + \|c\|_{L^p_{ul}(\Omega)}^{\frac{1}{2-(n/p)}}\right),$$

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then $u \equiv 0$.

Idea of proof: construct a signed solution of $\mathcal{L}[\psi] = 0$ in Ω.

Sharp Harnack implies ψ decays at most exponentially at infinity.

If u decayed quicker, we can can "squash" it with ψ, i.e. apply the comparison principle to $\pm u$ and $\delta\psi$ on the intersection of Ω with a large ball, for each $\delta > 0$.

Conclusion

We present a new approach which unifies the treatment of two classical problems, and has the following main advantages.

- It gives answers for operators with (even locally) unbounded lower-order coefficients, in a number of situations where all previous results required bounded ingredients.
- It extends many of the already available results on the Landis conjecture, even for equations with bounded coefficients; in particular, it proves the Landis conjecture for coercive fully nonlinear equations, a question which was completely open.
- It treats simultaneously equations in divergence and non-divergence form, and provides rather short proofs.

The main tools of our method are the weak and the full Harnack inequalities, with optimal dependence of their constants in the lower-order terms and the size of the domain.

Stronger form of Landis' conjecture

Landis conjecture

$$\mathcal{L}[u] = 0$$

implies for sufficiently large $M > 0$

$$\limsup_{|x|\to\infty} e^{M|x|}|u(x)| > 0.$$

but not true for $\liminf$,
even not true that

$$\int_{\partial B_R} |u| d\sigma \geq e^{-MR}$$

Conterexample $u(x) = e^{-x}\cos x$, which solves $u'' + 2u' + 2u = 0$.
Note that the latter operator does not satisfy the maximum principle on bounded subdomains, we have

$$\lambda_1(-L, \mathbb{R}) = -2.$$

Theorem (P. Souplet, B.S.)

Assume $\mathcal{L}_D$ satisfies the maximum principle in each bounded subdomain of Ω, AND has C^1-estimates. If

$$\mathcal{L}_D[u] = 0 \text{ in } \Omega, \quad \text{with } u \geq 0 \text{ on } \partial\Omega \text{ or } u \leq 0 \text{ on } \partial\Omega \text{ (if } \partial\Omega \text{ is not empty)}$$

then

$$\int_{\partial B_R} |u| d\sigma \geq e^{-MR} \int_{B_R} |u|$$

Theorem (P. Souplet, B.S.)

For every $\lambda > 0$, there exists an operator $\mathcal{L} := \Delta + b \cdot \nabla + c$ such that

$$\lambda_1(-\mathcal{L}, \mathbb{R}^n) = -\lambda$$

and the equation $\mathcal{L}u = 0$ admits a classical solution u on $\mathbb{R}^n$ such that, for some sequence $R_i \to \infty$ and some constant $M > 0$,

$$u(x) \equiv 0 \quad \text{on } |x| = R_i, \qquad \limsup_{|x| \to \infty} e^{M|x|} |u(x)| = 1.$$

Proof by a duality method, based on a ”boundary Harnack inequality” and Stampacchia maximum principle with optimized constants.

Theorem

Assume L has C^1-estimates. If $u \geq 0$ satisfies $\mathcal{L}u = g$ in B_R and $u = 0$ on ∂B_R, then

$$\sup_{B_R} \frac{u}{d} \leq e^{C_0 MR} \left(\inf_{B_R} \frac{u}{d} + \|g\|_{L^q_{ul}(B_R)} \right), \qquad \text{where } d = \mathrm{dist}(x, \partial B_R).$$

Theorem

If $\mathcal{L}_D$ satisfies the MP on bounded subdomains and has C^1-estimates, then the unique solution of the adjoint problem

$$\mathcal{L}_D^* z = f, \quad x \in B_R, \qquad z = 0, \quad x \in \partial B_R$$

is such that

$$\left\| \frac{z}{d} \right\|_\infty \leq \|f\|_q \, e^{-C_0 MR},$$

(set $q = \infty$, $f = sign(u)$), integrate, use the Divergence theorem...).

Further results

Lower bound on the principal eigenvalue: If $\mathcal{L}_D$ satisfies the MP on B_{2R}, then $\lambda_1(-\mathcal{L}_D, B_R) \geq -e^{C_0 MR}$.
Optimized quantitative Hopf lemma: If $-\mathcal{L}_D[u] \geq f \geq 0$, $u > 0$ in B_R, then

$$\inf_{B_1} \frac{u}{d} \geq e^{-C_0 MR} \int_{B_R} fd.$$

Optimized C^1-estimate: if $\mathcal{L}u = g$ in B_R and $u = 0$ on ∂B_R, then

$$\sup_{B_R} |\nabla u| \leq C_0 \left(M \sup_{B_R} |u| + M^{\frac{n}{q}-1} \|g\|_{L^p_{ul}(B_R)} \right).$$

Optimized log-grad estimate: if $\mathcal{L}u = 0$, $u > 0$ in B_1, then

$$\sup_{B_1} \frac{d|\nabla u|}{u} \leq C_0 M.$$

If $\mathcal{L}u = f \gneqq 0$, $u > 0$ in B_1, then

$$\sup_{B_1} \frac{d|\nabla u|}{u} \leq C_0 e^{C_0 M} \left(1 + \frac{\|f\|_q}{\|f\|_{1,d}} \right).$$

THANK YOU FOR LISTENING !