Overdetermined elliptic problems in the sphere

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Joint work with D. Ruiz and J. Wu

Cortona

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The problem. To understand the geometry of domains Ω that support a solution of the over-determined system

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega \end{cases}$$

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The most basic case is when Ω is a regular bounded domain of $\mathbb{R}^n$.

Serrin's theorem, 1971, ARMA.

If f is Lipschitz and Ω is a C^2 bounded domain where there exists a solution u to the problem

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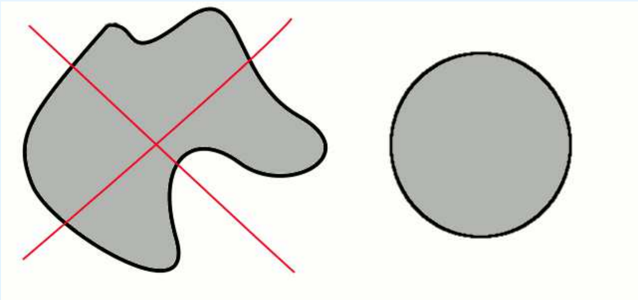

Moving plane method - Analogy with CMC surfaces

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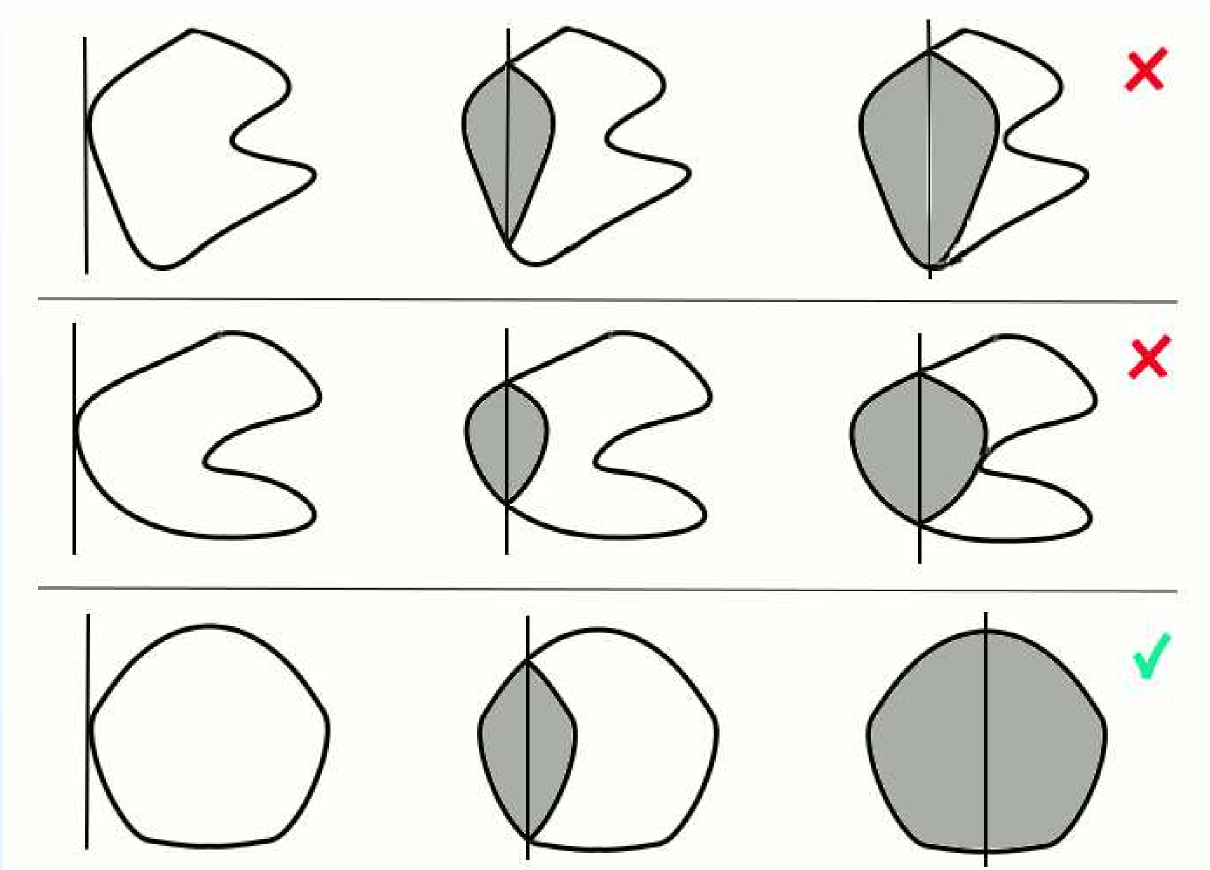

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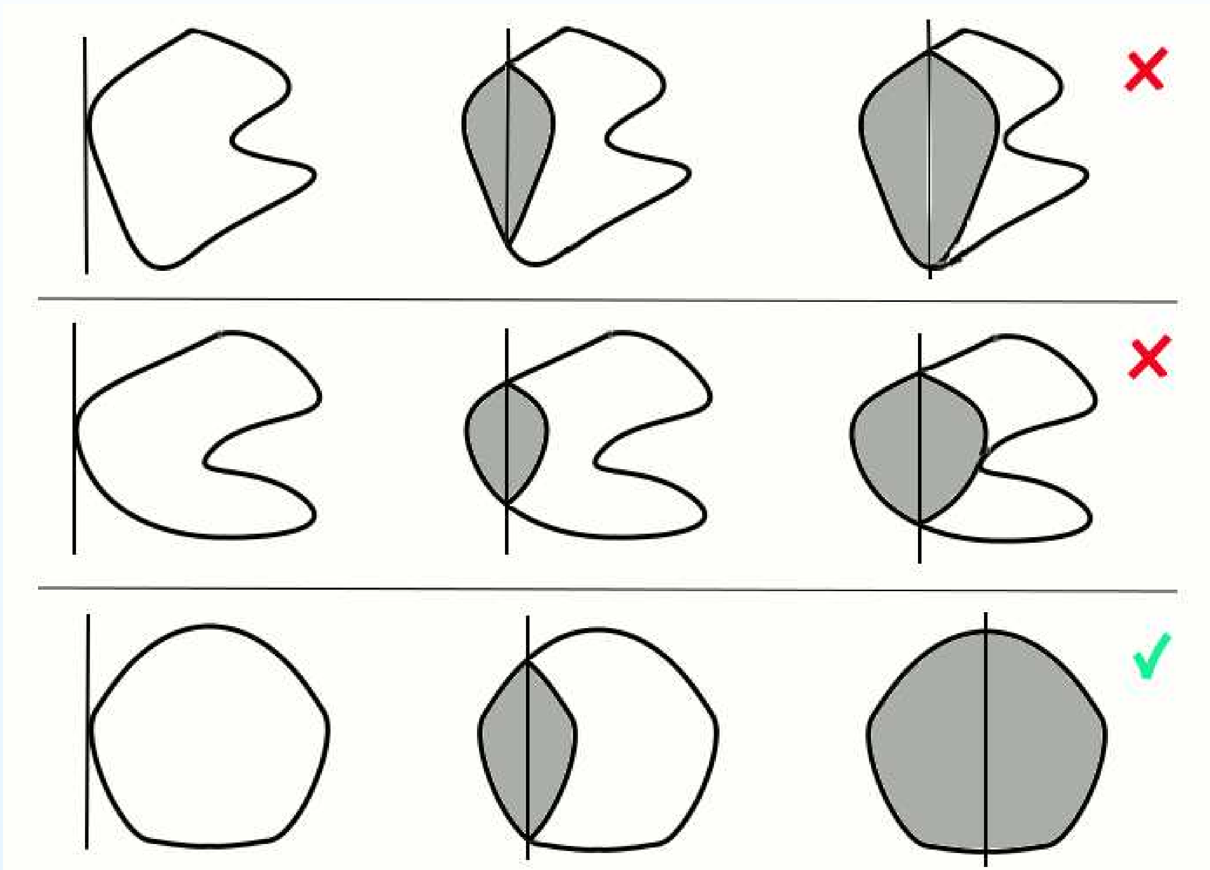

The proof of Serrin's moving plane method comes from the Alexandrof moving plane method, used to prove that the only compact **embedded** CMC surfaces are the spheres.

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The overdetermined problem now depends on the metric g:

$$\begin{cases} \Delta_g u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \nu_g} = \text{constant} & \text{on } \partial\Omega \end{cases}$$

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(4) Is it possible to obtain a Serrin's result for simply connected domains of the sphere?

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Let $\Omega \subset \mathbb{S}^2$ simply connected. Let $f : \mathbb{R}_+ \to \mathbb{R}$ verifying:

$$f \in C^1 \qquad , \qquad f(t) > 0 \qquad , \qquad f(t) \geq t\, f'(t)$$

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Proof: They generalize the method used by Hopf to prove that the only **immersed** genus 0 CMC surfaces in $\mathbb{R}^n$ are the spheres.

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And for a general f?

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Question (1997). Under the assumption that $\mathbb{R}^n \backslash \overline{\Omega}$ is connected and u is bounded, is it true that Ω must be a ball, or a half space, or a cylinder $\mathbb{R}^j \times B$ (where B is a ball) or the complement of one of these three domains?

Analogy with CMC surfaces

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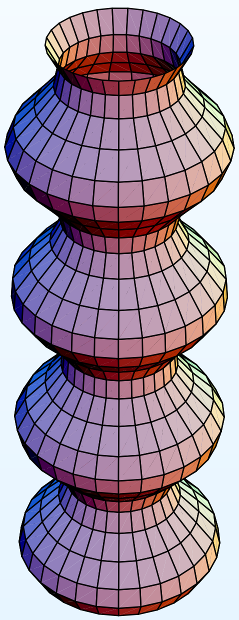
in domains whose boundary looks like Delaunay surfaces

$\Vert$



for $f(u) = \lambda u$ (S. 2010), or $f(u) = u - u^3$ (Del Pino, Pacard, Wei, 2015).

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(5) If Ω is the complement of a bounded region and u is a bounded solutions of

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then Ω must be the exterior of a ball (Reichel, 1997)

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Take a symmetry group G that leaves invariant the origin and, denoting by $\{\mu_{i_k}\}_{k\in\mathbb{N}}$ the eigenvalues of $\Delta_{\mathbb{S}^{n-1}}$ restricted to $G-$symmetric functions and m_k their multiplicity, require $i_1 \geq 2$ and m_1 odd (Example: $G = O(m) \times O(n-m)$, $m < n$)

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Theorem (Ros-Ruiz-Sicbaldi - JEMS 2020)

Let $1 < p < \frac{n+2}{n-2}$ ($p > 1$ if $n = 2$). There exist $R_* > 0$ such that the complement of the ball of radius R_* can be perturbed in non trivial G-symmetric domains Ω such that the problem

$$\begin{cases} -\Delta u + u - u^p = 0 & \text{in } \mathbb{R}^n \backslash \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega \end{cases}$$

admits a positive solution in $C^{2,\alpha} \cap H^1$.

Key ingredient

(Esteban - Lions, 1982) For any $R > 0$, there exists a radially symmetric C^∞ solution of

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We use the coordinates given by the exponential map centered at the south pole composed with polar coordinates in $\mathbb{R}^n$.

$$(r,\theta) \to \mathrm{exp}_S(r\,\theta) \qquad\qquad (r,\theta) \in \left[0, \frac{\pi}{\sqrt{k}}\right) \times \mathbb{S}^{n-1}$$

$$g_k = dr^2 + \frac{\sin^2(\sqrt{k}r)}{k} d\theta^2,$$

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$$\lim_{k \to 0} \|u_{k,\lambda} - \tilde{u}_\lambda\|_{H^1(\mathbb{S}^n(k)\backslash B_1)} = 0,$$

where $\tilde{u}_\lambda$ is the radial solution of the limit problem in $\mathbb{R}^n\backslash B_1$.

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There exists $\Lambda_0 > 0$ such that for any $\lambda > \Lambda_0$, for all $k \in (0, k_0)$ and for all function $v \in C^{2,\alpha}(\mathbb{S}^{n-1})$ whose norm is small, there exists a unique positive solution $u = u_k(\lambda, v) \in C^{2,\alpha}(\mathbb{S}^n(k) \backslash B_{1+v}) \cap H_0^1(\mathbb{S}^n(k) \backslash B_{1+v})$ to the problem

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where

$$B_{1+v} := \{(r, \theta) \in \mathbb{S}^n(k) \quad : \quad 0 \leq r < 1 + v(\theta)\} \ .$$

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$$B_{1+v} := \{(r, \theta) \in \mathbb{S}^n(k) \quad : \quad 0 \le r < 1 + v(\theta)\} \ .$$

In addition u depends smoothly on the function v, and $u = u_{k,\lambda}$ when $v \equiv 0$.

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For any $\lambda > \Lambda_0$, $k \in (0, k_0)$ and $v \in C^{2,\alpha}(\mathbb{S}^{n-1})$ with norm small, we define

$$F_k(\lambda, v) = \frac{\partial u_k(\lambda, v)}{\partial \nu} - \frac{1}{\mathrm{Vol}(\partial B_{1+v})} \int_{\partial B_{1+v}} \frac{\partial u_k(\lambda, v)}{\partial \nu}$$

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We show that there exists a bifurcation point $(\lambda_*(k), 0)$ for the equation $F_k(\lambda, 0) = 0$.

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THANK YOU FOR YOUR ATTENTION