Large time dynamics in nonlocal reaction-diffusion equations

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Mostly Maximum principle, Cortona, june 3, 2022

The question

$$\begin{aligned} u_t + u - K * u = &\ f(u) \ \ (t > 0, x \in \mathbb{R}) \\ u(0,x) = &\ u_0(x) \in [0,1], \ \ \mathrm{supp} u_0 \subset [-R, R]] \end{aligned}$$

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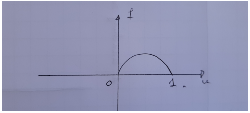

$$f: \; C^2, \; f'(u) \le f'(0).$$

$K(x)$: nonnegative, even, compactly supported, $\displaystyle\int_{\mathbb{R}} K = 1.$

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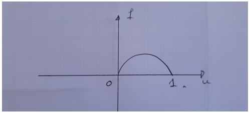

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Cauchy-Lipschitz + maximum principle

$\Longrightarrow$ Smooth $u(t,x) \in [0,1]$ (derivatives may grow exponentially).

Behaviour $t \to +\infty$?

Outline

- Main result.
- Occurrences of the model.
- Main steps of proofs of main results.

The main result

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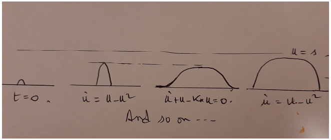
General picture

$$u(x) - K * u(x) = \int_{\mathbb{R}} K(x-y)\big(u(x)-u(y)\big)\,dy \text{ : Diffusion operator.}$$



$\Longrightarrow$ Invasion of unstable state 0 by stable state 1.

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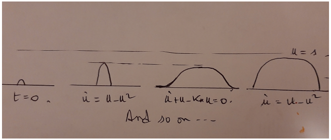

$\Longrightarrow$ Invasion of unstable state 0 by stable state 1.
$X(t)$: furthest point x to the right s.t. $u(t,x) = 1/2$.

Theorem

There are $c_* > 0$, $\lambda_* > 0$ *universal, and* x_∞ *depending on* $u(0)$ *s.t.*

$$X(t) = c_* t - \frac{3}{2\lambda_*} \ln\, t + x_\infty + o_{t\to+\infty}(1).$$

Who are c_* and λ_*?

- Heuristically, 0 "most unstable value of $u(t,x)$.

  $\Longrightarrow$ dynamics driven by small values.
- Linearised equation: $v_t + v - K * v = f'(0)v$
- Linear wave to the right:

  a solution $v(t,x) = e^{-\lambda(x-ct)}$, $\lambda > 0$, $c > 0$.

$$2\int_{\mathbb{R}} K(y)\big(\cosh(\lambda y) - 1\big)dy = c\lambda - f'(0).$$

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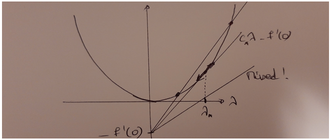

c_*: least $c > 0$ such that it is possible, λ_*: corresponding λ.

Motivations, occurrences of model

— Connexion w. diffusion of order 2

— Branching random walks

— Spatial spread of epidemics

Connexion w. second order diffusion

- $K(x)$: approximation of identity $K(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$.
- $\tau = \varepsilon^2 t$, $f(u) := \varepsilon^2 g(u)$.
- Expand in ε, throw away higher powers of ε.

$$v_\tau - dv_{xx} = g(v), \quad d = \frac{1}{2}\int_{\mathbb{R}} x^2\rho(x)dx.$$

The main results

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The main results

Theorem 1. (Kolmogorov, Petrovskii, Piskunov, 1937).
$X(\tau) = c_*\tau + o_{\tau\to+\infty}(\tau)$, with $c_* = 2\sqrt{dg'(0)}$.

Theorem 2. (Bramson, 1980-81). $f(u) = u - u^2$.

$$X(\tau) = c_*\tau - \frac{3}{2\lambda_*}\ln\,\tau + o_{\tau\to+\infty}(1), \text{ with } \lambda_* = \sqrt{\frac{d}{g'(0)}}.$$

- Logarithmic correction known as the *Bramson delay.*
- Bramson's proof relies on the study of rightmost particle in Branching Brownian motion.
- Deterministic proof provided by
  - Hamel, Nolen, Ryzhik, R. (2013, location of $X(t)$ up to $O(1)$ terms)
  - Nolen, Ryzhik, R. (2017, full Bramson theorem).

Branching random walks [ii]

On the real line $\mathbb{R}$:

- A particle initially sits at $x = 0$. Then
  - starts making jumps at random times.
  - At some time, splits in two.
  - Offsprings reproduce ancestor's behaviour.
- Law of random events:
  - Jumps and splitting times: Poisson distributions.
  - Jumps length: Density K.

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$Y(t)$: position of rightmost particle at time t.

$u(t,x)$**: probability that** $Y(t) \geq x$**.**

$$u_t + u - K * u = u - u^2, \quad u(0,x) = 1 - H(x).$$

(Mc Kean's representation formula)

Branching random walks [ii]

— If $X(t)$ is rightmost point s.t. $u(t,x) = 1/2$,

Study of $Y(t)$+McKean formula

$$\Longrightarrow X(t) = c_* t - \frac{3}{2\lambda_*}\ln\, t + x_\infty + o_{t\to+\infty}(t).$$

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Consequences

- Branching random walk approach solves the problem as soon as it comes from a Mc Kean's representation.
- Not all functions f, even concave ones, come from a McKean representation.

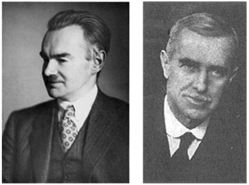
Earlier PDE work: Graham (2021), $X(t) = c_* t - \dfrac{3}{2\lambda_*}\ln\, t + O(1)$.

Models for spatial spread of epidemics: Homogeneous SI [i]

- $S(t)$: density of susceptibles at time t.
- $I(t)$: density of infectives at time t.

$$\begin{aligned} \dot{S} &= -\beta S I \\ \dot{I} &= \beta S I - \alpha I \\ S(0) &= S_0,\ I(0) = I_0 \text{ (usually} \ll 1) \end{aligned}$$

(A very particular case of) a model devised by Kermack and McKendrick (1927).

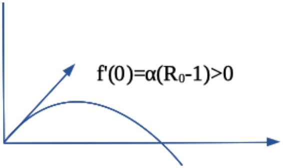


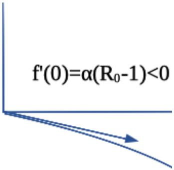
W. Kermack (1898-1970), A.G. McKendrick (1976-1943)

Homogeneous SI [ii]

Cumulated density of individuals: $u(t) = \int_0^t I(s)ds.$

$$\frac{d}{dt}\ln S = -\beta I \implies \dot{u} = S_0(1 - e^{-\beta u}) - \alpha u + I_0 := f(u) + I_0.$$





Define $R_0 = \dfrac{S_0\beta}{\alpha}.$

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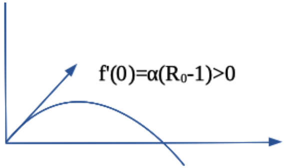

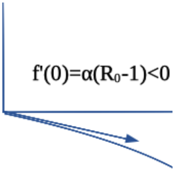

Define $R_0 = \frac{S_0\beta}{\alpha}.$

- $R_0 \leq 1$: epidemic will go extinct, $u(t) \to u_\infty(I_0)$ small.
- $R_0 > 1$: epidemics will spread, $u(t) \to u_\infty(I_0)$ of size independent of I_0. Susceptibles go down by $S_0 e^{-\beta u_\infty(I_0)}$.

Spatial effect: Nonlocal contaminations [i]

- **Assumption:** an infected is infectious within a certain range.
- One possibility: $\beta SI \to \beta S\ K * I$ (Kendall, 1956).

$$
\begin{array}{rl}
\partial_t S = -\beta S\ K * I,\ \partial_t I = & \beta S\ K * I - \alpha I \\
S(0,x) = S_0,\ \ I(0,x) = & I_0(x) \text{ small, comp. supported.}
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- Cumulated density of infected: $u(t,x) = \displaystyle\int_0^t I(s,x)ds.$

$$u_t = S_0(1 - e^{-\beta K * u}) - \alpha u + I_0.$$

- Nonlocal equation... but has a maximum principle!

(Monotone system).

Spatial effect: Nonlocal contamination [ii]

Theorem (Aronson, 1977). *$X(t)$: rightmost x s.t. $u(t,x)=\gamma$.*
THEN: $R_0>1 \implies X(t)=c_*t+o_{t\to+\infty}(t)$.

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- c_* computed from linearised equation
$$v_t+S_0\beta(v-K*v)=\alpha(R_0-1)v.$$
- Important subsequent theory: more elaborate models, abstract monotone systems theory...
- Sharp time asymptotics?
  - Epidemiological relevance can be questionned, but mathematical question in its own right.
- Our approach
  - works for Kendall's model.
  - Gives information about I and S not available before.

Proof of main result: Main steps

— Travelling waves
— The tail of the solution
— Adjusting a travelling wave to the solution

Travelling waves

Travelling wave w. speed c**:** $u(t,x) = \varphi(x - ct)$.

$$\varphi - K * \varphi - c\varphi' = f(\varphi), \quad \varphi(-\infty) = 1, \ \varphi(+\infty) = 0.$$

For every $c \geq c_*$, *there is a unique wave profile* φ_c *w. speed* c.

(Diekman 1979,..., Coville 2003, Carr-Chmaj 2004)

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1st attempt: KPP's original idea for $u_t - u_{xx} = u - u^2$:

- $t \mapsto u_x(t,.)$ increases along a level curve of u.
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Why? *What you're seeing and what you're reading is not what's happening.* (D. Trump, 2018)

The tail of the solution

Run with speed c_***:** $x := x - c_* t$. Set $u(t,x) = e^{-\lambda_* x} v(t,x)$:

$$v_t + \mathcal{I}_* v + e^{-\lambda_* x} v^2, \qquad \mathcal{I}_* v = v - e^{-\lambda_* x} K * v - c_* \partial_x v.$$

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Main statement: as $t \to +\infty$ and $t^\delta \le x \le t^{1/2+\delta}$ we have ($\delta > 0$ small)

$$u(t,x) \sim \quad \frac{\alpha_\infty x}{t^{3/2}} e^{-\lambda_* x - \frac{x^2}{4 d_* t}}$$

$$\alpha_\infty > 0 : \text{ depends on } \quad \text{initial datum}, \quad d_* = \int_{\mathbb{R}} x^2 e^{-\lambda_* x} K(x) dx.$$

Main issue: no regularising effect.

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Hint why this may be true: we have

$$e^{-t\mathcal{I}_*} v_0(x) = e^{t d_* \partial_{xx}} v_0(x) + O(e^{-t^\gamma}).$$

Where the logarithmic term comes from

Travelling wave at infinity:
$\varphi_{c_*}(x) = (x + k_*)e^{-\lambda_* x} + O\big(e^{-(\lambda_* + \gamma)x}\big).$
(shared feature with usual diffusion)

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Translate φ_{c_*} by $\sigma(t)$ to match u at $x = t^\delta$:

$$\frac{\alpha_\infty}{t^{3/2}} e^{-\lambda_* x - \frac{x^2}{4d_* t}} = (x + \sigma(t) + k_*)e^{-\lambda_*(x+\sigma(t))} \text{ at } x = t^\delta$$

$$\Longrightarrow \sigma(t) = \frac{1}{\lambda_*}\Big(\frac{3}{2t} - \ln \alpha_\infty + o(1)\Big)$$

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Theorem. *We have $u(t,x) \sim_{t\to+\infty} \varphi_{c_*}\big(x + \sigma(t)\big)$.*

Proof. Write a BVP for for $u(t,x) - \varphi_{c_*}\big(x + \sigma(t)\big)$ on $(-\infty, t^\delta)$, use that it has to be controlled on a domain of size $\sim t^\delta$.

Ongoing work

- **Discrete Fisher-KPP** (w. C. Besse, G. Faye and M. Zhang).
- **Fisher-KPP in periodic environments** (w. A. Novikov and L. Ryzhik, earlier work w. F. Hamel, J. Nolen and L. Ryzhik).
- **Coupled diffusion/SI models on networks** (w. G. Faye and M. Zhang).

Thank you for attention