Periodic homogenization of the principal eigenvalue of second-order elliptic operators

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Introduction

Setting: the principal eigenvalue problem

Under suitable assumptions on $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ there exists $(\phi_1, \lambda_1) \in C_+(\Omega) \times \mathbb{R}$ solving

$$\begin{cases} F(x, \phi_1, D\phi_1, D^2\phi_1) & = -\lambda_1 \phi_1, \quad \text{in } \Omega \\ \phi_1 & = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

In particular, we focus on the *principal eigenvalue* $\lambda_1 = \lambda_1^+(F, \Omega)$, characterized by

$$\lambda_1^+(F, \Omega) = \sup\{\lambda : \exists\, \phi > 0 \text{ in } \Omega,\ F(x, \phi, D\phi, D^2\phi) \leq -\lambda\phi \text{ in } \Omega\},$$

and is *unique* and *simple* ([Quaas and Sirakov, 2008], [Armstrong, 2009], following [Berestycki et al., 1994]).

Setting: periodic homogenization

We consider, for $\epsilon \in (0,1)$, the eigenvalue problem

$$F(x, x/\epsilon, u, Du, D^2u) = -\lambda^\epsilon u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad (\mathrm{EV}^\epsilon)$$

Setting: periodic homogenization

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- Our aim is the study of stability results for the principal eigenvalue problem in the context of periodic homogenization:
- to characterize the limit of solutions u^ϵ as $\epsilon \to 0$, and
- to obtain a rate of convergence to this limit state, i.e., $\|u^\epsilon - u\|_\infty \leq \omega(\epsilon)$, for some explicit modulus of continuity ω.
- We follow the viscosity solutions approach contained in the classical works [Lions et al., 1986], [Evans, 1989], [Evans, 1992].

Motivation: periodic homogenization and dynamics

Consider

$$\begin{cases} \dot{x} = b(x, x/\epsilon, \alpha), & t > 0, \\ x(0) = x_0 \in \Omega, \end{cases}$$

where $\epsilon \in (0, 1)$, $b : \Omega \times \mathbb{T}^N \times \mathcal{A} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain, $\mathbb{T}^N$ the N-dimensional flat torus.

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Defining $y = {}^{x}\!/\!_{\epsilon}$, we have

$$\dot{y} = \epsilon^{-1}\dot{x} = \epsilon^{-1} b(x, {}^{x}\!/\!_{\epsilon}, \alpha) = \epsilon^{-1} b(x, y, \alpha).$$

Thus,

$$\begin{cases} \dot{x} = b(x, y, \alpha), & t > 0, \\ \dot{y} = \epsilon^{-1} b(x, y, \alpha), & t > 0, \\ x(0) = x_0 \in \Omega, & y(0) = {}^{x_0}\!/\!_{\epsilon} \in \mathbb{R}^N. \end{cases}$$

Here we see two components of a system evolving at *different time scales.*

This translates to the PDE framework via the *Dynamical Programming Principle.*

Assumptions I

Let S^N denote the set of $N \times N$ symmetric matrices.

Assume $F \in C(\Omega \times \mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ satisfies the uniform ellipticity condition

$$\begin{aligned} & M^{-}_{\lambda,\Lambda}(Y) - C_1(|q| + |s|) \\ & \quad \leq F(x, y, r + s, p + q, X + Y) - F(x, y, r, p, X) \\ & \quad \leq M^{+}_{\lambda,\Lambda}(Y) + C_1(|q| + |s|), \end{aligned} \tag{A1}$$

for some $0 < \lambda \leq \Lambda < +\infty$ and $C_1 > 0$, for all $x \in \Omega$, $y \in \mathbb{T}^N$, $r, s \in \mathbb{R}$, $p, q \in \mathbb{R}^N$ and $X, Y \in S^N$.

Assumptions II

We also assume that F is positively 1-homogeneous in its last three variables,

$$F(x, y, \alpha r, \alpha p, \alpha X) = \alpha F(x, y, r, p, X). \tag{A2}$$

From the previous assumptions on F, for each $\epsilon \in (0, 1)$, there exists a *principal eigenvalue* for the operator $F^\epsilon \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N)$ defined by

$$F^\epsilon(x, r, p, X) = F(x, {}^{x}\!/\!_{\epsilon}, r, p, X),$$

that is, the existence of a pair $(u^\epsilon, \lambda^\epsilon) \in C(\bar{\Omega}) \times \mathbb{R}$ with $u^\epsilon > 0$ in Ω solving (EV^ϵ) with λ^ϵ satisfying the same extremal characterization.

Homogenization, the viscosity solutions approach

For each x, r, p, X, there exists a unique $c \in \mathbb{R}$ for which the *cell problem*

$$F(x, y, r, p, D^2_{yy}v(y) + X) = c \quad \text{in } \mathbb{T}^N, \tag{CP}$$

has a viscosity solution.

This can be obtained formally through the ansatz for (EV$^\epsilon$):

$$u^\epsilon(x) = u(x) + \epsilon^2 v(x/\epsilon).$$

Define $\bar{F} : \Omega \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ as

$$\bar{F}(x, r, p, X) = c,$$

where $c = c(x, r, p, X)$ is the unique constant for which (CP) has a solution, known as the *effective Hamiltonian*.

The effective problem

It can be shown that $\bar{F}$ "inherits" the properties assumed for the original F, therefore

$$\bar{F}(x, u, Du, D^2u) = -\lambda u \ \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{EV}$$

is solvable as well.

We denote by $\lambda(\bar{F})$ the principal eigenvalue associated to positive eigenfunctions, which as before is unique and simple.

In short, *the eigenvalue problem is "averaged", or homogenized, by the same operator as the stationary one.*

Previous results, proper setting I

In the *proper* setting, i.e., when

$$u \mapsto F(x, y, r, p, X) - \mu r \quad \text{is nonincreasing for some} \quad \mu > 0$$

and we consider

$$F(x, {}^{x}/_{\epsilon}, u^{\epsilon}, Du^{\epsilon}, D^2 u^{\epsilon}) = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

and

$$\bar{F}(x, u, Du, D^2 u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In [Evans, 1992], it is shown that $u^{\epsilon} \to u$ unif. over Ω;

Previous results, proper setting II

Regarding the rate of convergence,

- in [Camilli and Marchi, 2009], for

$$u(x) + F(x, x/\epsilon, Du^\epsilon, D^2 u^\epsilon) = 0 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

  with F is convex/concave, $\|u - u^\epsilon\|_\infty \leq C\epsilon^\beta$, for some $\beta > 0$ depending on the data.
- The assumpion of convexity/concavity is crucial in the use of $C^{2,\alpha}$ estimates for u and the associated (approximate) corrector.
- For F with simpler structure, we have improved rates:

$$\text{if } F = F(x/\epsilon, D^2 u^\epsilon), \text{then } \|u - u^\epsilon\| \leq C\epsilon^2;$$

$$\text{if } F = F(x/\epsilon, Du^\epsilon, D^2 u^\epsilon), \text{ then } \|u - u^\epsilon\| \leq C\epsilon;$$

Previous results, proper setting III

- In [Capuzzo-Dolcetta and Ishii, 2001], analogous results are obtained for

$$u^\epsilon(x) + F(x, {}^x\!/_\epsilon, Du^\epsilon) = 0 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

  but instead of relying on regularity we have a careful doubling of variables argument.

- In [Kim and Lee, 2016] we find higher order-order expansions and rates of convergence for both linear and nonlinear equations, assuming sufficient smoothness for F.

Previous results, eigenvalue problems

- In the setting of compact operators in Banach spaces, in [Osborn, 1975] the author obtains different bounds for the *gap* between the subspaces generated by eigenfunctions and their limit—in an application, this gives a quadratic rate of convergence for the eigenvalues.
- Building on these ideas, in [Kesavan, 1979] the author obtains a convergence result for the entire spectrum of oscillating self-adjoint operators in divergence form.
- Rates of convergence are given in terms of an auxiliary problem whose solution serves as a pivot between u^ϵ and u (or their analogue).

Results

Convergence result (homogenization)

Theorem (Dávila–R.-P.–Topp, *preprint*)

Assume F satisfies (A1), (A2) *and* $(u^\epsilon, \lambda^\epsilon)$ *is the principal solution pair of* (EV$^\epsilon$). *Then,*

$$\lambda^\epsilon \to \lambda(\bar{F}) \text{ as } \epsilon \to 0.$$

If we consider appropriately normalized u^ϵ (e.g. $\|u^\epsilon\|_\infty = 1$), then

$$u^\epsilon \to u \text{ as } \epsilon \to 0, \text{ uniformly on } \bar{\Omega}$$

where u solves (EV) *and* $\|u\|_\infty = 1$.

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$$u^\epsilon \to u \text{ as } \epsilon \to 0, \text{ uniformly on } \bar{\Omega}$$

where u solves (EV) *and* $\|u\|_\infty = 1$.

- The proof follows the classical *perturbed test function* method of [Evans, 1989], coupled with the tools of [Berestycki et al., 1994].

Rate of convergence for the principal eigenvalue

Regarding the rate of convergence for the eigenvalues, we focus on

$$\begin{cases} F\left(\frac{x}{\epsilon}, D^2 u^\epsilon\right) = -\lambda^\epsilon u^\epsilon & \text{in } \Omega, \\ u^\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{nLin$^\epsilon$}$$

and

$$\begin{cases} \bar{F}(D^2 u) = -\bar{\lambda} u & \text{in } \Omega, \\ \bar{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{nLin}$$

We can prove the following rate of convergence for $\{\lambda^\epsilon\}_\epsilon$ under additional assumptions on F.

Rate of convergence for the principal eigenvalue

Theorem (Dávila–R.-P.–Topp, *preprint*)

Assume $F \in C^{4,1}(\mathbb{T}^N \times S^N)$ *is convex in its second variable, and satisfies* (A1), (A2). *Let* $\lambda^\epsilon, \bar{\lambda}$ *denote the principal eigenvalues associated to* (nLin^ϵ) *and* (nLin), *respectively. Then, there exists a constant* C *depending only on* F *and* Ω *such that*

$$|\lambda^\epsilon - \bar{\lambda}| \leq C\epsilon.$$

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- The proof relies on the construction of higher-order correctors as in [Kim and Lee, 2016] and the variational formula for the principal eigenvalue of [Donsker and Varadhan, 1976].

Rates of convergence for the principal eigenfunction

Given $a : \mathbb{T}^N \to S^N$, $b : \mathbb{T}^N \to \mathbb{R}^N$, and $c : \mathbb{T}^N \to \mathbb{R}$, we define

$$L^\epsilon u(x) = a(x/\epsilon) D^2_{xx} u(x) + b(x/\epsilon) \cdot D_x u(x) + c(x/\epsilon) u(x),$$

where $aM = \operatorname{tr}\left(aM\right)$, and consider

$$L^\epsilon u^\epsilon = -\lambda^\epsilon u^\epsilon \quad \text{in } \Omega, \qquad u^\epsilon = 0 \quad \text{on } \partial\Omega. \tag{Lin$^\epsilon$}$$

Rates of convergence for the principal eigenfunction

By the homogenization result we have that the solution pair $(u^\epsilon, \lambda^\epsilon)$ converges to the solution pair $(u, \lambda(\bar{L}))$ of

$$\bar{L}u = -\lambda(\bar{L})u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{Lin}$$

where $\lambda(\bar{L}) \in \mathbb{R}$ denotes the principal eigenvalue associated to

$$\bar{L}u(x) = \bar{a}D^2_{xx}u + \bar{b} \cdot D_x u + \bar{c}u,$$

for some constant, uniquely defined $\bar{a} \in S^N$, $\bar{b} \in \mathbb{R}^N$, and $\bar{c} \in \mathbb{R}$.

Rates of convergence for the principal eigenfunction

Theorem (Dávila–R.-P.–Topp, *preprint*)

Assume $a \in C^6(\mathbb{T}^N; S^N)$, $b \in C^6(\mathbb{T}^N; \mathbb{R}^N)$, *and* $c \in C^6(\mathbb{T}^N)$ *in* (Lin$^\epsilon$). *Let* u *be a solution to* (Lin) *with* $u > 0$ *in* Ω. *Then, there exists* $C > 0$ *depending on the coefficients* a, b, c *and* Ω, *such that, for all* $\epsilon \in (0, 1)$, *there exists* u^ϵ *solving* (Lin$^\epsilon$) *with* $u^\epsilon > 0$ *in* Ω, *satisfying*

$$|\lambda^\epsilon - \lambda| + \|u^\epsilon - u\|_{L^\infty(\Omega)} \leq C\epsilon.$$

Rates of convergence for the principal eigenfunction

Theorem (Dávila–R.-P.–Topp, *preprint*)

Assume $a \in C^6(\mathbb{T}^N; S^N)$, $b \in C^6(\mathbb{T}^N; \mathbb{R}^N)$, and $c \in C^6(\mathbb{T}^N)$ in (Lin$^\epsilon$)*. Let u be a solution to* (Lin) *with $u > 0$ in Ω. Then, there exists $C > 0$ depending on the coefficients a, b, c and Ω, such that, for all $\epsilon \in (0, 1)$, there exists u^ϵ solving* (Lin$^\epsilon$) *with $u^\epsilon > 0$ in Ω, satisfying*

$$|\lambda^\epsilon - \lambda| + \|u^\epsilon - u\|_{L^\infty(\Omega)} \leq C\epsilon.$$

- For the rate of the eigenfunction, the result can be obtaned as an application of [Osborn, 1975]; we provide a proof exclusively by PDE techniques, employing also the strategy of [Kesavan, 1979].
- The rate depends on the “correct” normalization of solutions.

Sketches of the proofs

General convergence result (homogenization)

- $\{\lambda^\epsilon\}$ is uniformly bounded: from the characterization

$$\lambda^\epsilon = \sup\left\{\lambda \mid \exists u > 0, F^\epsilon(x, u, Du, D^2u) \geq -\lambda u \text{ in } \Omega\right\}$$

  and particular barrier functions, we can show $-C_1 \leq \lambda^\epsilon \leq CR^{-2}$, where C, C_1 are independent of ϵ and R is such that $B_R \subset \Omega$ (see [Berestycki et al., 1994]).
- $\{u^\epsilon\}$ is precompact: has u^ϵ, C^α estimates depending only on the L^∞ norm of the right-hand side of (Lin$^\epsilon$),

$$\|\lambda^\epsilon u^\epsilon\|_\infty = |\lambda^\epsilon|\|u^\epsilon\|_\infty = |\lambda^\epsilon|$$

General convergence result (homogenization)

- Let $\bar{\lambda} = \limsup_{\epsilon \to 0} \lambda^\epsilon$; assume $\lambda^\epsilon \to \bar{\lambda}$ (keeping ϵ for the subsequence), and $u^\epsilon \to \bar{u}$ for some $\bar{u} \in C(\Omega)$.
- Following [Evans, 1992], we can obtain that $\bar{u}$ is a viscosity solution of
$$\bar{F}(x, \bar{u}, D\bar{u}, D^2\bar{u}) \geq -\bar{\lambda}\bar{u} \quad \text{in } \Omega.$$
- By the SMP, $\bar{u} > 0$; hence, by the extremal characterization, $\bar{\lambda} \leq \lambda(\bar{F})$.
- By similar arguments, $\lambda(\bar{F}) \leq \underline{\lambda} = \liminf_{\epsilon \to 0} \lambda^\epsilon$, hence $\lambda(\bar{F}) = \bar{\lambda} = \underline{\lambda}$, and thus we have the convergence of the eigenvalues.
- Each *cluster point* of $\{u^\epsilon\}_\epsilon$ is a positive eigenfunction corresponding to $\lambda(\bar{F})$, and by the *simplicity* of $\lambda(\bar{F})$, we conclude.

Rates of convergence, higher-order correctors I

- A key element of the proof is the construction of *higher-order correctors*, as in [Kim and Lee, 2016].
- We address the case of (Lin$^\epsilon$) for simplicity, in which we use a third-order expansion:

$$v^\epsilon(x) = \epsilon w_1(x) + \epsilon^2(w_2(x, {}^x\!/\!_\epsilon) + z_2^\epsilon(x)) + \epsilon^3(w_3(x, {}^x\!/\!_\epsilon) + z_3^\epsilon(x)),$$

  where the w_k, $k = 1, 2, 3$, are interior correctors and the z_k^ϵ correct the behavior of w_k at the boundary, i.e.,

$$L^\epsilon z_k^\epsilon = 0 \quad \text{in } \Omega; \qquad z_k^\epsilon(x) = -w_k(x, x/\epsilon) \quad \text{on } \Omega.$$

- In particular,

$$w_2(x, {}^x\!/\!_\epsilon) = v({}^x\!/\!_\epsilon; x, u(x), Du(x), D^2u(x)),$$

  the standard (*second-order*) corrector.

Rates of convergence, higher-order correctors II

- Given $k, l = 1, \ldots, N$, we consider

$$a(y)D^2_{yy}\chi(y) + a_{kl}(y) = \gamma, \quad y \in \mathbb{T}^N,$$

  where (χ, γ) is an ergodic pair. It is easy to show that $\gamma = \bar{a}_{kl}$, the corresponding entry of the diffusion matrix in (Lin), and we write $\chi = \chi^{kl}$.

- Ignoring lower order-terms (for simplicity), we define our second-order corrector as

$$w_2(x, y) = \chi^{kl}(y)\partial^2_{kl}u(x), \tag{1}$$

- Given $k, l, m = 1, \ldots, N$, we denote by $(\chi = \chi^{klm}, \bar{a}_{klm})$ the unique pair solving

$$a(y)D^2_{yy}\chi(y) + 2a_{*m}(y) \cdot D_y\chi^{kl}(y) = \bar{a}_{klm} \quad \text{in } \mathbb{T}^N, \tag{2}$$

  where a_{*m} denotes the m-th column of a.

Rates of convergence, higher-order correctors III

- We then define $w_1(x, x/\epsilon) = \psi_1(x)$ as the unique solution to

$$\begin{cases} \bar{L}\psi_1 = -\bar{a}_{klm}\partial^3_{klm}u(x) & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

- In turn, we define the third-order corrector as

$$w_3(x, y) = \chi^{klm}(y)\partial^3_{klm}u(x) + \chi^{kl}(y)\partial^2_{kl}\psi_1(x)$$

In the end, we obtain

- A *recursive* equation:

$$a(y)D^2_{xx}\psi_1(x) + 2a(y)D^2_{yx}w_2 + a(y)D^2_{yy}w_3 = 0, \quad y \in \mathbb{T}^N.$$

- w_k, z_k are uniformly bounded in terms of ϵ;
- in particular, $\|v^\epsilon\|_\infty \leq C\epsilon$ for some $C > 0$;
- w_k is at least $C^{2,\alpha}$ in both x and y, uniformly in ϵ.

Rates of convergence, higher-order correctors IV

We can repeat the process for each lower-order term:

- Given $k = 1, \dots N$, $\bar{b}_k$ and $\bar{c}$ in (Lin) we solve the ergodic problems

$$a(y)D^2_{yy}\eta(y) + b_k(y) = \bar{b}_k, \quad y \in \mathbb{T}^N,$$

  whose solution is denoted by η^k, and

$$a(y)D^2_{yy}\nu(y) + c(y) = \bar{c}, \quad y \in \mathbb{T}^N,$$

  whose solution is denoted simply by ν.
- We can normalize so that $\eta^k \equiv 0$ if $b \equiv 0$ and $\nu \equiv 0$ if $c \equiv 0$.

Rates of convergence, higher-order correctors V

For nonlinear F, the second-order term is obtained as before,

$$w_2(x, y) := w(y; D^2_{xx} u(x)), \quad x \in \Omega, \; y \in \mathbb{T}^N,$$

and we proceed by linearizing F at $w_2(y; D^2_{xx} u(x)) \ldots$

Rates of convergence for the eigenvalues

- (One half of) the estimate follows from substituting an expansion of u^ϵ based on v^ϵ (denoted $\tilde{v}^\epsilon$) into the classical *minimax formula* of [Donsker and Varadhan, 1976]: for some probability measure on Ω, $d\mu^\epsilon = d\mu^\epsilon(x)$,

$$\begin{aligned}\lambda - \lambda^\epsilon = \lambda + \inf_{\phi > 0} \int_\Omega \frac{L^\epsilon \phi(x)}{\phi(x)} d\mu^\epsilon &\leq \lambda + \int_\Omega \frac{L^\epsilon \tilde{v}^\epsilon(x)}{\tilde{v}^\epsilon(x)} d\mu^\epsilon \\ &= \int_\Omega \frac{(L^\epsilon + \lambda)\tilde{v}^\epsilon(x)}{\tilde{v}^\epsilon(x)} d\mu^\epsilon \leq \ldots \leq C\epsilon\end{aligned}$$

Rates of convergence for the eigenfunctions I

Key idea is to consider, for u the solution to (Lin), $\|u\|_\infty = 1$, $\epsilon \in (0, 1)$,

$$\begin{cases} L^\epsilon w^\epsilon = -\lambda u & \text{in } \Omega \\ w^\epsilon = 0 & \text{in } \partial\Omega, \end{cases} \tag{P^ϵ}$$

Replacing λ^ϵ and λ with $\lambda^\epsilon + C_1 + 1$, $\lambda + C_1 + 1$, resp., we can assume L^ϵ and $\bar{L}$ are both *proper*.

Also notice that $\bar{L}$ is "still" the effective Hamiltonian associated to (P^ϵ).

By the results in [Kim and Lee, 2016], $\|w^\epsilon - u\|_\infty \leq C\epsilon$, for some $C > 0$ independent of ϵ.

Rates of convergence for the eigenfunctions II

We define $z^\epsilon = u^\epsilon - w^\epsilon + t_\epsilon u^\epsilon$, choosing $t^\epsilon \in \mathbb{R}$ such that $(z^\epsilon, u^\epsilon) = 0$;
z^ϵ solves

$$\begin{cases} L^\epsilon z^\epsilon + \lambda^\epsilon z^\epsilon & = -\lambda^\epsilon w^\epsilon + \lambda u \quad \text{in } \Omega, \\ z^\epsilon & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4}$$

from which we can show (via a *blow-up* argument) that

$$\|z^\epsilon\|_\infty \leq C_0 \epsilon, \tag{5}$$

but this is

$$\|(1 + t^\epsilon)u^\epsilon - w^\epsilon\|_\infty \leq C_0 \epsilon,$$

i.e., the *precise normalization* to obtain the rate.

Grazie!

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