On large solutions for fractional Hamilton-Jacobi equations

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Large solution for nonlocal Hamilton-Jacobi equation

We are interested in existence of large (unbounded) solutions of (model problem)

$$(*) \quad \begin{cases} (-\Delta)^s u + |Du|^p + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

for $0 < p < 2s, \quad f \in C(\Omega) \cap L^\infty(\Omega)$, with

$$\lim_{x \to \partial\Omega, x \in \Omega} u(x) = +\infty \quad \text{or} \quad \lim_{x \to \partial\Omega, x \in \Omega} u(x) = -\infty,$$

where

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(z)}{|x - z|^{N+2s}} dz,$$

Large solutions

Keller [CPAM'57], and Osserman [PJM'57] study large solutions associated to reaction-diffusion problems with the form

$$-\Delta u = f(u) \quad \text{in } \Omega.$$

They proved the existence of large solutions if

$$\int_0^\delta \frac{ds}{\sqrt{F(s)}} < \infty \quad F' = f,$$

For instance, if $f(u) = u^p$, we have existence of large solutions iff $p > 1$.

Obs: Connection with probability (superdiffusions) in the regime $1 < p \leq 2$, see Le Gall's book '99

Lasry-Lions 1989: Existence of large solution

They studied the problem

$$-\Delta u + |Du|^p + \lambda u = f \quad \text{in } \Omega,$$

for $1 < p \leq 2$, $\lambda > 0$ and $f \in C(\Omega)$. They prove that there exists a unique solution u that behaves like

$$d(x)^{-\frac{p-2}{p-1}}$$

near the boundary, where

$$d(x) := \text{dist}(x, \partial\Omega).$$

and with logarithmic profile in the critical case $p = 2$.
The case $p > 2$ is also discuss in that work! (Bounded solution).

Lasry-Lions 1989

Connection with optimal control of a stochastic differential equation (constrain a Brownian by controlling its drift):

- Admisible drift are so that the process never exit the domain Ω,
- f is part of the running cost,
- λ discount factor.

Using the dynamic programming principle they prove that a value function is the unique solution of the equation.
This value function is obtained by minimizing running cost (in the class of admissible drift) involving f and “feedback" term depending on p and the drift term.

Ergodic problem

The ergodic problem is the limit of solution $u_\lambda - u_\lambda(x_0) \to v$ as $\lambda \to 0$ that is a large solution v of

$$-\Delta v + |Dv|^p = f + c_0 \quad \text{in } \Omega,$$

where c_0 is the ergodic constants and $v(x_0) = 0$.

Connection with parabolic problem

$$u_t - \Delta u + |Du|^p = f \quad \text{in } \Omega \times (0, +\infty) \qquad \text{(VHJ-B)}$$

$$u = g \quad \text{on } \partial\Omega \times (0, +\infty) \qquad \text{(BC)}$$

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in } \overline{\Omega}. \qquad \text{(IC)}$$

Barles-Porretta-Tchamba 2010.
They proved:

- Further properties of the ergodic problem (type of eigenvalue problem and characterization of c_0).
- If $c_0 < 0$ then stationary problem has a solution and u(x,t) converge to that solution.
- If the stationary problem has no solution then ergodic constants satisfies $c_0 \geq 0$

Connection with parabolic problem

- Established that if $c_0 > 0$ and $3/2 < p \leq 2$

$$u(x,t) + c_0 t \to v_0 + \mu \quad \text{as} \quad t \to \infty$$

  locally uniform in Ω, for some constant μ.
- Other asymptotic results in the rest of the cases ($c_0 = 0$ or/and $1 < p < 3/2$).

- Tchamba 2010 studied the case $p > 2$.

Case of the parabolic problem in $\mathbb{R}^n$

- Barles Meireles 2016 (uniqueness of Ergodic problem) $p > 2$ +regularity of f. simplicity even for subsolution (generalize uniqueness).
- Arapostathis, Biswas, Caffarelli (2019). (Uniqueness of Ergodic problem) $1 < p \leq 2$ for solution.
- Previous results by Ichihara 2012 -Barles Meireles 2016 (polynomial growth)

Case of the parabolic problem in $\mathbb{R}^n$

Barles-Q.-Rodríguez (2021)-Q. Rodríguez (2022)

Very general u_0 and f (arbitrary growth) and for any $p > 1$

$$u(x,t) + c_0 t \to v_0 + \mu,$$

for (c_0, v_0) the ergodic par in $\mathbb{R}^n$.

Key ingredients:

Case $p > 2$. Unbounded super-solution that move (in time) the boundary of explosion to have the first convergence: half relax limit is a sub-solution of the ergodic problem.

Case $1 < p \leq 2$ generalize simplicity even for sub-solution.

Some previous results on large solution in nonlocal setting

For equation of the type

$$(-\Delta)^s u = f(u, x)$$

in a bounded domain.

- Abatangelo (2015-2017) (Properties of Green function)
- Chen-Felmer-Q (2015) (Close the our approach: Perron type method)

Main results (model case)

$$(*) \quad \begin{cases} (-\Delta)^s u + |Du|^p + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

We write $p_i = p_i(s)$ for $i = 1, 2$ as

$$p_1 = s + \frac{1}{2} \quad \text{and} \quad p_2 = \frac{s+1}{2-s}, \tag{1}$$

A third exponent $p_0 = \frac{2s}{2-s}$ We notice that for $s \in (1/2, 1)$, we have $p_1 > 1$ and

$$p_1(s) < p_2(s) < 2s, \quad p_2(1^-) = 2, \quad p_1(1/2^+) = p_2(1/2^+) = 1,$$

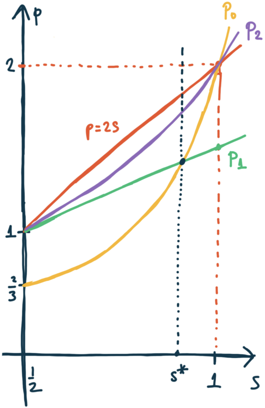

Figure: *Exponents* p_0, p_1 *and* p_2 *as a function of* s. *Notice that* p_0 *and* p_1 *intersects at* $s^* = \frac{\sqrt{17}-1}{4}$.

Main results (model case and $\lambda \geq 0$)

Theorem: Let $s \in (1/2, 1)$, $0 < p < 2s$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, $f \in L^\infty(\Omega) \cap C(\Omega)$. When $1 < p < 2s$, we denote

$$\beta := (2s - p)/(1 - p) < 0.$$

1.- One parameter family of solutions (close to s-harmonic): If $0 < p < p_2$, there exist a family of solutions $\{u_t\}_{t \in \mathbb{R}, t \neq 0}$ to $(*)$, such that for each t we have

$$d^{1-s} u_t(x) - t = O(d^\gamma),$$

for some $\gamma > s - 1$ depending on p. In particular, if $t_1 < t_2$, then

$$u_{t_1} < u_{t_2} \quad \text{in } \Omega.$$

Moreover, if p additionally satisfies $p < p_0$, then $\gamma > 0$.

Main results (model case and $\lambda \geq 0$)

2.- Positive scale solution: If $p_1 < p < p_2$, then there exists a constant $T > 0$ and a function u solving $(*)$ such that

$$d(x)^{-\beta}u(x) - T = O(d(x)^{\gamma}),$$

for some $\gamma > 0$.

3.- Negative scale solution: For $p_2 < p < 2s$, then there exist $T > 0$ and a solution u of $(*)$ such that

$$d^{-\beta}(x)u(x) + T = O(d(x)^{\gamma}),$$

for some $\gamma > 0$.

Remarks

- Case 1 and Case 2 can occur simultaneously (non-uniqueness) and we have $u_t < u$ in Ω.
- Condition $p_1 < p$ is such that $d^\beta \in L^1(\Omega)$ and the nonlocal operator is well define.

Ideas of the proof (one dimensional case)

$$(-\Delta)^s x_+^\tau = -c(\tau) x_+^{\tau - 2s}, \quad x > 0$$

here $(\cdot)_+$ denote positive part. where $c(\tau)$ is convex function and has the form.

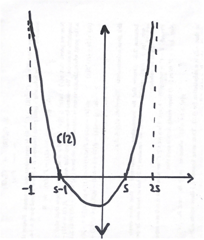

Figure: $c(\tau)$

In the local case:

$$(-\Delta) x_+^\tau = -(\tau(\tau - 1)) x_+^{\tau - 2},$$

Lemma

Let $\Omega \subset \mathbb{R}^N$ *be a bounded domain with* C^2 *boundary,* $s \in (0,1)$*. Then, for each* $\tau \in (-1, 2s)$*, we have*

$$(-\Delta)^s d^\tau(x) = -d^{\tau-2s}(x)(c(\tau) + O(d(x)^s)), \quad \textit{close to de boundary}$$

where

$$c(\tau) = \text{P.V.} \int_{\mathbb{R}} [(1+z)_+^\tau - 1]|z|^{-(1+2s)} dz.$$

This constant (of the one dimensional case) and therefore $c(-1^+) = +\infty$, $c(2s^-) = +\infty$, $c(s-1) = c(s) = 0$, $c(\tau) > 0$ *if* $\tau \in (-1, s-1) \cup (s, 2s)$ *and* $c(\tau) < 0$ *for* $\tau \in (s-1, s)$*.*

Condition $p_1 < p < p_2$ is such that $c(\beta) > 0$ and is posible to construct sub and super solution to the problem (Larsy-Lions type solution) and applied a Perron type method.

Some extension

For a class of fully nonlinear nonlocal operator $\mathcal{I}$ with kernels comparable to the fraction Laplacian.

$$\begin{cases} -\mathcal{I}u + |Du|^p + \lambda u & = f \qquad \text{in } \Omega, \\ u & = \varphi \qquad \text{in } \Omega^c, \\ \lim\limits_{x\in\Omega,\ x\to\partial\Omega} u & = +\infty, \end{cases} \tag{2}$$

and its blow-down version $\lim\limits_{x\in\Omega,\ x\to\partial\Omega} u = -\infty$.

here $\varphi \in L^1_\omega(\bar{\Omega}^c)$, ($\omega$ is so that $\mathcal{I}$ is well define); $\lambda > -\lambda_0(\mathcal{I})$ (splitting $\mathcal{I}$ into the censored problem and the rest, with $\lambda_0(\mathcal{I}) > 0$), In the model case $(-\mathcal{I} = (-\Delta)^s)$

$$\lambda_0 = \inf_{x\in\Omega} \int_{\Omega^c} |x-z|^{-(N+2s)} dz.$$

Thanks for the attention!