Coupling and doubling and timely decay

Alessio Porretta
University of Rome Tor Vergata

Mostly Maximum Principle
Cortona, May 30-June 3, 2022

Coupling and doubling and timely decay
Isn't what nature has meant us to play ?

[E. Kean: "Shakespeare's flowers", 1833]

Fokker-Planck equations: trend to equilibrium

A classical issue: analyse the convergence of solutions of FP equations in $\mathbb{R}^d$ towards the unique stationary invariant measure

$$\begin{cases} \partial_t m + Lm - \text{ div } (b(x)m) = 0 \\ m(0) = m_0 \, , \int_{\mathbb{R}^d} m_0 = 1 \end{cases} \quad \overset{t\to\infty}{\longrightarrow} \quad \bar{m} \, : \quad \begin{cases} L(\bar{m}) = \text{ div } (b\bar{m}) \, , \\ \int_{\mathbb{R}^d} \bar{m} = 1 \end{cases}$$

where L is a diffusion process (local or nonlocal).

Fokker-Planck equations: trend to equilibrium

A classical issue: analyse the convergence of solutions of FP equations in $\mathbb{R}^d$ towards the unique stationary invariant measure

$$\begin{cases} \partial_t m + Lm - \operatorname{div}(b(x)m) = 0 \\ m(0) = m_0\,, \int_{\mathbb{R}^d} m_0 = 1 \end{cases} \quad \overset{t\to\infty}{\longrightarrow} \quad \bar m\ : \quad \begin{cases} L(\bar m) = \operatorname{div}(b\bar m)\,, \\ \int_{\mathbb{R}^d} \bar m = 1 \end{cases}$$

where L is a diffusion process (local or nonlocal).

Related to ergodicity of the associated stochastic process X_t:

$$\underbrace{\frac{1}{T}\mathbb{E}\int_0^T \varphi(X_t)dt}_{=\frac{1}{T}\int_0^T\int_{\mathbb{R}^d}\varphi dm(t,x)} \overset{T\to\infty}{\to} \int_{\mathbb{R}^d}\varphi\, d\bar m \qquad \forall \varphi \in C_c(\mathbb{R}^d)$$

Model example is the Ornstein-Ulhenbeck process in $\mathbb{R}^d$.

$$dX_t = -X_t dt + \sqrt{2}dB_t \qquad \to \partial_t m - \Delta m - \operatorname{div}(xm) = 0$$

$\rightsquigarrow$ FP equations with confining drift

Rate of convergence ⇝ time decay analysis

Rephrase the problem as time decay for zero average distributions:

$$\begin{cases} \partial_t \mu + L\mu - \operatorname{div}\,(b(t,x)\mu) = 0 \\ \mu(0) = \mu_0\,,\ \int_{\mathbb{R}^d} \mu_0 = 0 \end{cases} \quad \to \quad \|\mu(t)\|_X \overset{t\to\infty}{\to} 0$$

Rate of convergence $\rightsquigarrow$ time decay analysis

Rephrase the problem as time decay for zero average distributions:

$$\begin{cases} \partial_t \mu + L\mu - \text{ div } (b(t,x)\mu) = 0 \\ \mu(0) = \mu_0 \, , \int_{\mathbb{R}^d} \mu_0 = 0 \end{cases} \qquad \to \qquad \|\mu(t)\|_X \overset{t\to\infty}{\to} 0$$

Main focus: L can be a Levy operator

$$Lv = -\mathrm{tr}(Q(x)D^2 v) - B\cdot Dv - \int_{\mathbb{R}^d} \{v(x+z) - v(x) - (Dv(x)\cdot z)1_{|z|\le 1}\}\nu(dz)$$

Model case: the fractional Laplacian $\nu = \frac{dz}{|z|^{d+\alpha}}$

Rate of convergence $\rightsquigarrow$ time decay analysis

Rephrase the problem as time decay for zero average distributions:

$$\begin{cases} \partial_t \mu + L\mu - \operatorname{div}\,(b(t,x)\mu) = 0 \\ \mu(0) = \mu_0\,,\ \int_{\mathbb{R}^d} \mu_0 = 0 \end{cases} \quad \to \quad \|\mu(t)\|_X \overset{t\to\infty}{\to} 0$$

Main focus: L can be a Levy operator

$$Lv = -\mathrm{tr}(Q(x)D^2 v) - B\cdot Dv - \int_{\mathbb{R}^d} \{v(x+z) - v(x) - (Dv(x)\cdot z)1_{|z|\le 1}\}\nu(dz)$$

Model case: the fractional Laplacian $\nu = \frac{dz}{|z|^{d+\alpha}}$

Motivation: Long time behavior of Nash equilibria in Mean-field game theory

In that context: b depends on the individual strategies
$\rightsquigarrow$ $b(t,x) = H_p(x, Du)$, where u is the value function of the agents

Rate of convergence $\rightsquigarrow$ time decay analysis

Rephrase the problem as time decay for zero average distributions:

$$\begin{cases} \partial_t \mu + L\mu - \text{ div } (b(t,x)\mu) = 0 \\ \mu(0) = \mu_0 \, , \int_{\mathbb{R}^d} \mu_0 = 0 \end{cases} \quad \rightarrow \quad \|\mu(t)\|_X \overset{t\to\infty}{\to} 0$$

Main focus: L can be a Levy operator

$$Lv = -\mathrm{tr}(Q(x)D^2 v) - B\cdot Dv - \int_{\mathbb{R}^d} \{v(x+z) - v(x) - (Dv(x)\cdot z)1_{|z|\leq 1}\}\nu(dz)$$

Model case: the fractional Laplacian $\nu = \frac{dz}{|z|^{d+\alpha}}$

Motivation: Long time behavior of Nash equilibria in Mean-field game theory

In that context: b depends on the individual strategies
$\rightsquigarrow$ $b(t,x) = H_p(x, Du)$, where u is the value function of the agents

Main point: $b(t,x)$ is not very regular, it is time-dependent, it is not well-known a priori, etc...
$\rightsquigarrow$ need a very robust study of FP equation

Typical results for $\begin{cases} \partial_t m - \Delta m - \text{ div } (b(x)m) = 0 \\ \int m_0 = 0 \end{cases}$

- Rate is exponential if $b(x) \cdot x \geq |x|^2$ for $|x| \to +\infty$ (es. Ornstein-Uhlenbeck semigroup)

$$\|m(t)\|_X \leq e^{-\omega t}\|m_0\|_X$$

Typical results for $\begin{cases} \partial_t m - \Delta m - \text{ div } (b(x)m) = 0 \\ \int m_0 = 0 \end{cases}$

- Rate is exponential if $b(x) \cdot x \geq |x|^2$ for $|x| \to +\infty$ (es. Ornstein-Uhlenbeck semigroup)

$$\|m(t)\|_X \leq e^{-\omega t} \|m_0\|_X$$

- Rate is slower if $b(x) \cdot x \geq |x|^\gamma$ for $|x| \to +\infty$, with $\gamma \in (0, 2)$ (slowly confining drifts $\rightsquigarrow$ sub-geometrical rate)

Typical results for $\begin{cases} \partial_t m - \Delta m - \text{ div } (b(x)m) = 0 \\ \int m_0 = 0 \end{cases}$

- Rate is exponential if $b(x) \cdot x \geq |x|^2$ for $|x| \to +\infty$ (es. Ornstein-Uhlenbeck semigroup)

$$\|m(t)\|_X \leq e^{-\omega t}\|m_0\|_X$$

- Rate is slower if $b(x) \cdot x \geq |x|^\gamma$ for $|x| \to +\infty$, with $\gamma \in (0,2)$ (slowly confining drifts $\rightsquigarrow$ sub-geometrical rate)

- Natural choice of X is a L^1-weighted space:

$$\text{es:} \quad X = L^1(\langle x \rangle^\kappa)\,, \qquad \langle x \rangle = \sqrt{1+|x|^2}$$

This implies decay in L^1-norm for $m(t)$ but requires some finite moments on the initial data m_0.

A (very partial !) look at the (very huge !) literature $\rightsquigarrow$ two major axes

1. Approach by energy/entropy methods

Typical target:

$$\partial_t m \underbrace{-\Delta m - \operatorname{div}(\nabla V\, m)}_{Lm} = 0$$

A (very partial !) look at the (very huge !) literature $\rightsquigarrow$ two major axes

1. Approach by energy/entropy methods

Typical target:

$$\partial_t m \underbrace{-\Delta m - \operatorname{div}(\nabla V\, m)}_{Lm} = 0$$

- Spectral gap in $L^2(e^V)$
  [Gross '75, Liggett '91, Rockner-F.Y. Wang '01...semigroup theory...

A (very partial !) look at the (very huge !) literature $\rightsquigarrow$ two major axes

1. Approach by energy/entropy methods

Typical target:

$$\partial_t m \underbrace{-\Delta m - \operatorname{div}(\nabla V\, m)}_{Lm} = 0$$

- Spectral gap in $L^2(e^V)$
  [Gross '75, Liggett '91, Rockner-F.Y. Wang '01...semigroup theory...
- entropy dissipation

$$\frac{d}{dt} H(m|e^{-V}) \le -\gamma\, H(m|e^{-V})\,, \qquad H(m|e^{-V}) := \int m(\log m + V)dx$$

[Toscani, Villani, Markovich, Carrillo...]

A (very partial !) look at the (very huge !) literature $\rightsquigarrow$ two major axes

1. Approach by energy/entropy methods

Typical target:

$$\partial_t m \underbrace{-\Delta m - \operatorname{div}(\nabla V\, m)}_{Lm} = 0$$

- Spectral gap in $L^2(e^V)$
  [Gross '75, Liggett '91, Rockner-F.Y. Wang '01...semigroup theory...
- entropy dissipation

$$\frac{d}{dt} H(m|e^{-V}) \leq -\gamma\, H(m|e^{-V}), \qquad H(m|e^{-V}) := \int m(\log m + V)dx$$

  [Toscani, Villani, Markovich, Carrillo...]
- Extensions to fractional Laplacian: [Biler-Karch '03, Tristani '15, Gentil-Imbert '09, ...]

Key tools are functional inequalities: Poincaré inequality (strong and weak forms), log-Sobolev inequality... (Gross, Bakry-Emery, Villani..)

1. Approach by probabilistic methods/ideas

[Meyn &Tweedie '93, Douc-Fort-Guillin '09, Hairer, Hairer-Mattingly '11, ...]

Typical setting:

$$m(t) = e^{tA} m_0$$

where A is a diffusion process.

1. Approach by probabilistic methods/ideas

[Meyn &Tweedie '93, Douc-Fort-Guillin '09, Hairer, Hairer-Mattingly '11, ...]

Typical setting:

$$m(t) = e^{tA} m_0$$

where A is a diffusion process.

- $\exists$ of a Lyapunov function + local strict positivity of the semigroup $\rightsquigarrow$ exponential decay. In rough terms:

$$\begin{cases} \exists\, \varphi(x) \; : \; A^* \varphi \geq \gamma \varphi - C\, 1_K \,, \; K \text{ compact} \\ m(t,x) \geq \nu > 0 \quad \forall x \in K \end{cases} \Rightarrow \|m(t)\|_X \leq C\, e^{-\omega t} \|m_0\|_X$$

where X is a L^1-weighted space (depends on Lyapunov function φ).

1. Approach by probabilistic methods/ideas

[Meyn &Tweedie '93, Douc-Fort-Guillin '09, Hairer, Hairer-Mattingly '11, ...]

Typical setting:

$$m(t) = e^{tA} m_0$$

where A is a diffusion process.

- $\exists$ of a Lyapunov function + local strict positivity of the semigroup $\rightsquigarrow$ exponential decay. In rough terms:

$$\begin{cases} \exists\, \varphi(x) \;:\; A^*\varphi \geq \gamma\varphi - C\, 1_K \,,\; K \text{ compact} \\ m(t,x) \geq \nu > 0 \quad \forall x \in K \end{cases} \Rightarrow \|m(t)\|_X \leq C\, e^{-\omega t}\|m_0\|_X$$

  where X is a L^1-weighted space (depends on Lyapunov function φ).

1. Approach by probabilistic methods/ideas

[Meyn &Tweedie '93, Douc-Fort-Guillin '09, Hairer, Hairer-Mattingly '11, ...]

Typical setting:

$$m(t) = e^{tA} m_0$$

where A is a diffusion process.

- $\exists$ of a Lyapunov function + local strict positivity of the semigroup $\rightsquigarrow$ exponential decay. In rough terms:

$$\begin{cases} \exists\, \varphi(x) \, : \, A^*\varphi \geq \gamma\varphi - C\, 1_K \, , \, K \text{ compact} \\ m(t,x) \geq \nu > 0 \quad \forall x \in K \end{cases} \Rightarrow \|m(t)\|_X \leq C\, e^{-\omega t} \|m_0\|_X$$

  where X is a L^1-weighted space (depends on Lyapunov function φ).

- Further works also used a mix of ingredients (Lyapunov + Poincaré, decomposition methods... See e.g. [Bakry-Cattiaux-Guillin JFA '08], [Mischler, Mouhot-Mischler '09, Kavian-Mischler-Ndao '21], [LaFleche'20],...

We suggest a simplification/generalization of some of the previous ideas

We suggest a simplification/generalization of some of the previous ideas

- By duality, the decay of FP equations is entirely deduced from dissipation estimates of drift-diffusion equations

We suggest a simplification/generalization of some of the previous ideas

- By duality, the decay of FP equations is entirely deduced from dissipation estimates of drift-diffusion equations

- The core lies in new weighted oscillation estimates for

$$\begin{cases} \partial_t u + L^* u + b(t,x)\cdot Du = 0 \\ u(0) = u_0 \,, \end{cases}$$

We define the seminorm

$$[u]_\varphi := \sup_{x,y\in\mathbb{R}^{2d}} \frac{|u(x)-u(y)|}{\varphi(x)+\varphi(y)}$$

where typically $\varphi(x)$ is a Lyapunov function.

We suggest a simplification/generalization of some of the previous ideas

- By duality, the decay of FP equations is entirely deduced from dissipation estimates of drift-diffusion equations
- The core lies in new weighted oscillation estimates for

$$\begin{cases} \partial_t u + L^* u + b(t,x) \cdot Du = 0 \\ u(0) = u_0 \,, \end{cases}$$

We define the seminorm

$$[u]_\varphi := \sup_{x,y \in \mathbb{R}^{2d}} \frac{|u(x) - u(y)|}{\varphi(x) + \varphi(y)}$$

where typically $\varphi(x)$ is a Lyapunov function.
Special case: $\varphi(x)$ is of power-type:

$$[u]_{\langle x \rangle^k} = \sup_{x,y \in \mathbb{R}^{2d}} \frac{|u(x) - u(y)|}{\langle x \rangle^k + \langle y \rangle^k} \,, \qquad \langle x \rangle = \sqrt{1 + |x|^2} \,.$$

Sample result: fractional Laplacian + drift

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u + b(t,x) \cdot Du = 0 & t > 0 \\ u(0) = u_0 \,, \end{cases}$$

Theorem (A)

Assume that

$$b(t,x) \cdot x \geq \lambda \, |x|^2 \qquad \forall (x,t) \,:\, |x| \text{ is large}$$

Under either of the following conditions:

(i) $\alpha \in (1,2]$ *and* $(b(t,x) - b(t,y)) \cdot (x-y) \geq -c_0|x-y|$, *for all* t, x, y

(ii) $\alpha \in (0,1]$ *and*
$(b(t,x) - b(t,y)) \cdot (x-y) \geq -c_0|x-y|(1 \wedge |x-y|^{1-\alpha+\delta})$,
for some $\delta > 0$, $c_0 > 0$,

then there exist K, ω*:*

$$[u(t)]_{\langle x \rangle^k} \leq K \, e^{-\omega t} \, [u_0]_{\langle x \rangle^k}$$

Theorem (B)

Under the same conditions of Theorem (A), the solution of the FP equation

$$\begin{cases} \partial_t m + (-\Delta)^{\alpha/2} m - \ div\,(b(t,x)m) = 0 \\ \mu(0) = m_0\,,\ \int_{\mathbb{R}^d} m_0 = 0 \end{cases}$$

decays exponentially:

$$\|m(t)\|_{L^1(\langle x\rangle^\kappa)} \leq K\, e^{-\omega t}\, \|m_0\|_{L^1(\langle x\rangle^k)}$$

Theorem (B)

Under the same conditions of Theorem (A), the solution of the FP equation

$$\begin{cases} \partial_t m + (-\Delta)^{\alpha/2} m - \ div\,(b(t,x)m) = 0 \\ \mu(0) = m_0\,, \int_{\mathbb{R}^d} m_0 = 0 \end{cases}$$

decays exponentially:

$$\|m(t)\|_{L^1(\langle x \rangle^\kappa)} \le K\, e^{-\omega t}\, \|m_0\|_{L^1(\langle x \rangle^k)}$$

Rmk: the conditions on the drift are $b(t,x)\cdot x \ge |x|^2$ for $|x| \to \infty$ and

(i) $\alpha \in (1,2]$ and $(b(t,x) - b(t,y))\cdot(x-y) \ge -c_0|x-y|$,

(ii) $\alpha \in (0,1]$ and, for some $\delta > 0$,
$(b(t,x) - b(t,y))\cdot(x-y) \ge -c_0|x-y|(1 \wedge |x-y|^{1-\alpha+\delta})$

- for $\alpha \in (1,2]$ (elliptic case), minimal requirements on $b(t,x)$: "confining at infinity + locally bounded".

Theorem (B)

Under the same conditions of Theorem (A), the solution of the FP equation

$$\begin{cases} \partial_t m + (-\Delta)^{\alpha/2} m - \ div\,(b(t,x)m) = 0 \\ \mu(0) = m_0\,,\ \int_{\mathbb{R}^d} m_0 = 0 \end{cases}$$

decays exponentially:

$$\|m(t)\|_{L^1(\langle x\rangle^\kappa)} \leq K\,e^{-\omega t}\,\|m_0\|_{L^1(\langle x\rangle^k)}$$

Rmk: the conditions on the drift are $b(t,x)\cdot x \geq |x|^2$ for $|x| \to \infty$ and

(i) $\alpha \in (1,2]$ and $(b(t,x) - b(t,y))\cdot(x-y) \geq -c_0|x-y|$,

(ii) $\alpha \in (0,1]$ and, for some $\delta > 0$,
$(b(t,x) - b(t,y))\cdot(x-y) \geq -c_0|x-y|(1 \wedge |x-y|^{1-\alpha+\delta})$

- for $\alpha \in (1,2]$ (elliptic case), minimal requirements on $b(t,x)$: "confining at infinity + locally bounded".
- for $\alpha \in (0,1]$, some local Hölder regularity is required on the (non dissipative part of) drift $b(t,x)$.

Proof of Thm (B) from Thm (A):

- By duality we have

$$\int_{\mathbb{R}^d} \xi\, m(t)dx = \int_{\mathbb{R}^d} m_0\, u(0)dx$$

$$\forall \xi, u\ :\ \begin{cases} -\partial_t u + (-\Delta)^{\alpha/2} u + b(t,x)\cdot Du = 0 & \text{in } (0,t) \\ u(t) = \xi\,, \end{cases}$$

Proof of Thm (B) from Thm (A):

- By duality we have

$$\int_{\mathbb{R}^d} \xi\, m(t)dx = \int_{\mathbb{R}^d} m_0\, u(0)dx$$

$$\forall \xi, u \ : \ \begin{cases} -\partial_t u + (-\Delta)^{\alpha/2} u + b(t,x)\cdot Du = 0 & \text{in } (0,t) \\ u(t) = \xi\,, \end{cases}$$

- Using m_0 with zero average we have, $\forall c \in \mathbb{R}$

$$\int_{\mathbb{R}^d} \xi\, m(t)dx = \int_{\mathbb{R}^d} m_0\, u(0)dx = \int_{\mathbb{R}^d} m_0\,(u(0)+c)dx \qquad (1)$$
$$\leq \|u(0)+c\|_{L^\infty(\langle x\rangle^{-k}dx)} \|m_0\|_{L^1(\langle x\rangle^k)}$$

Proof of Thm (B) from Thm (A):

- By duality we have

$$\int_{\mathbb{R}^d} \xi \, m(t)dx = \int_{\mathbb{R}^d} m_0 \, u(0)dx$$

$$\forall \xi, u \ : \ \begin{cases} -\partial_t u + (-\Delta)^{\alpha/2} u + b(t,x) \cdot Du = 0 & \text{in } (0,t) \\ u(t) = \xi \, , \end{cases}$$

- Using m_0 with zero average we have, $\forall c \in \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}^d} \xi \, m(t)dx = \int_{\mathbb{R}^d} m_0 \, u(0)dx = \int_{\mathbb{R}^d} m_0 \, (u(0) + c)dx \\ \leq \|u(0) + c\|_{L^\infty(\langle x \rangle^{-k} dx)} \|m_0\|_{L^1(\langle x \rangle^k)} \end{aligned} \tag{1}$$

- We use equivalence of seminorms

$$\inf_{c \in \mathbb{R}} \|u + c\|_{L^\infty(\langle x \rangle^{-k} dx)} = \sup_{x,y \in \mathbb{R}^{2d}} \frac{|u(x) - u(y)|}{\langle x \rangle^k + \langle y \rangle^k} = [u]_{\langle x \rangle^k}$$

Proof of Thm (B) from Thm (A):

- By duality we have

$$\int_{\mathbb{R}^d} \xi\, m(t)dx = \int_{\mathbb{R}^d} m_0\, u(0)dx$$

$$\forall \xi, u \;:\; \begin{cases} -\partial_t u + (-\Delta)^{\alpha/2} u + b(t,x)\cdot Du = 0 & \text{in } (0,t) \\ u(t) = \xi\,, \end{cases}$$

- Using m_0 with zero average we have, $\forall c \in \mathbb{R}$

$$\int_{\mathbb{R}^d} \xi\, m(t)dx = \int_{\mathbb{R}^d} m_0\, u(0)dx = \int_{\mathbb{R}^d} m_0\,(u(0)+c)dx \qquad (1)$$
$$\le \|u(0)+c\|_{L^\infty(\langle x\rangle^{-k}dx)} \|m_0\|_{L^1(\langle x\rangle^k)}$$

- We use equivalence of seminorms

$$\inf_{c\in\mathbb{R}} \|u+c\|_{L^\infty(\langle x\rangle^{-k}dx)} = \sup_{x,y\in\mathbb{R}^{2d}} \frac{|u(x)-u(y)|}{\langle x\rangle^k + \langle y\rangle^k} = [u]_{\langle x\rangle^k}$$

- We minimize on c in (1) and use Thm (A)

$$\int_{\mathbb{R}^d} \xi\, m(t)dx \le \|m_0\|_{L^1(\langle x\rangle^k)} [u(0)]_{\langle x\rangle^k} \le \|m_0\|_{L^1(\langle x\rangle^k)}\, K e^{-\omega t} \|\xi\|_{L^\infty(\langle x\rangle^{-k}dx)}$$

We prove similar results for the *slowly confining case*:

$$b(t,x)\cdot x \geq c\,|x|^{\gamma} \qquad \forall (x,t)\;:\;|x| \text{ is large} \tag{2}$$

whenever

$$\gamma \in (0,2)$$

Theorem

Assume that b satisfies (2) with $\gamma \in (2-\alpha, 2)$ and b satisfies the conditions of Theorem (A). Let m be the solution of the FP equation

$$\begin{cases} \partial_t m + (-\Delta)^{\alpha/2} m - \; div\,(b(t,x)m) = 0 \\ \mu(0) = m_0\,,\; \int_{\mathbb{R}^d} m_0 = 0\,. \end{cases}$$

Then, for any $k \in (2-\gamma, \alpha)$ and $\bar{k} > k$, we have

$$\|m(t)\|_{L^1(\langle x\rangle^k)} \leq K\,(1+t)^{-q}\,\|m_0\|_{L^1(\langle x\rangle^{\bar{k}})} \qquad \text{where } q = \frac{\bar{k}-k}{2-\gamma}.$$

Main novelty of this approach: decay of FP equations $\Longleftrightarrow$ decay in weighted seminorms for drift-diffusion eqs

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u + b(t,x) \cdot Du = 0 & t > 0 \\ u(0) = u_0 \,, \end{cases}$$

$$\rightsquigarrow \quad [u(t)]_{\langle x \rangle^k} \leq K\, e^{-\omega t}\, [u_0]_{\langle x \rangle^k}$$

Main novelty of this approach: decay of FP equations $\Longleftrightarrow$ decay in weighted seminorms for drift-diffusion eqs

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u + b(t,x)\cdot Du = 0 & t>0 \\ u(0) = u_0\,, \end{cases}$$

$$\rightsquigarrow \quad [u(t)]_{\langle x\rangle^k} \leq K\, e^{-\omega t}\, [u_0]_{\langle x\rangle^k}$$

Rephrasing:

$$\exists\ \omega, K>0\ :\ u(t,x)-u(t,y) \leq K\, e^{-\omega t}\ \left(\langle x\rangle^k + \langle y\rangle^k\right) \qquad (3)$$

Main novelty of this approach: decay of FP equations $\Longleftrightarrow$ decay in weighted seminorms for drift-diffusion eqs

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u + b(t,x) \cdot Du = 0 & t > 0 \\ u(0) = u_0 \,, \end{cases}$$

$$\rightsquigarrow \quad [u(t)]_{\langle x \rangle^k} \leq K\, e^{-\omega t} \, [u_0]_{\langle x \rangle^k}$$

Rephrasing:

$$\exists \; \omega, K > 0 \; : u(t,x) - u(t,y) \leq K\, e^{-\omega t} \, \left(\langle x \rangle^k + \langle y \rangle^k \right) \tag{3}$$

Main idea: (3) is OK if we prove

$$u(t,x) - u(t,y) \leq e^{-\omega t} \, \left\{ K[\langle x \rangle^k + \langle y \rangle^k] + \psi(|x-y|) \right\} \tag{4}$$

for some bounded $\psi(\cdot)$.

Main novelty of this approach: decay of FP equations $\iff$ decay in weighted seminorms for drift-diffusion eqs

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u + b(t,x)\cdot Du = 0 & t > 0 \\ u(0) = u_0 \,, \end{cases}$$

$$\rightsquigarrow \quad [u(t)]_{\langle x\rangle^k} \leq K\, e^{-\omega t}\, [u_0]_{\langle x\rangle^k}$$

Rephrasing:

$$\exists\, \omega, K > 0 \,:\, u(t,x) - u(t,y) \leq K\, e^{-\omega t}\, \left(\langle x\rangle^k + \langle y\rangle^k\right) \qquad (3)$$

Main idea: (3) is OK if we prove

$$u(t,x) - u(t,y) \leq e^{-\omega t}\, \left\{K[\langle x\rangle^k + \langle y\rangle^k] + \psi(|x-y|)\right\} \qquad (4)$$

for some bounded $\psi(\cdot)$.

- $\psi(|x-y|)$ takes care of *small range interactions*
  Typically: ψ is a concave bounded function which is locally Hölder
- *long range interactions* only happen at infinity
  $\rightsquigarrow$ dominated by the Lyapunov function

Decay of weighted seminorms:

$$u(t,x) - u(t,y) \leq e^{-\omega t} \left\{ K \underbrace{\left[\langle x\rangle^k + \langle y\rangle^k\right]}_{\text{Lyapunov}} + \underbrace{\psi(|x-y|)}_{\text{local ellipticity}} \right\} \quad (5)$$

- Estimate (5) is an evidence of ergodicity of the underlying process.

  Similar results exist in the probabilistic literature in the form of *contraction estimates for transition probabilities in Wasserstein's metrics* [F.Y. Wang], [Eberle], [Schilling-J.Wang], [Majka]...

Decay of weighted seminorms:

$$u(t,x) - u(t,y) \leq e^{-\omega t} \left\{ K \underbrace{\left[\langle x\rangle^k + \langle y\rangle^k\right]}_{\text{Lyapunov}} + \underbrace{\psi(|x-y|)}_{\text{local ellipticity}} \right\} \qquad (5)$$

- Estimate (5) is an evidence of ergodicity of the underlying process.

  Similar results exist in the probabilistic literature in the form of *contraction estimates for transition probabilities in Wasserstein's metrics* [F.Y. Wang], [Eberle], [Schilling-J.Wang], [Majka]...
- Our proof is entirely analytic and quite elementary: max. principle + doubling variables' method

Decay of weighted seminorms:

$$u(t,x) - u(t,y) \leq e^{-\omega t} \left\{ K \underbrace{\left[\langle x \rangle^k + \langle y \rangle^k\right]}_{\text{Lyapunov}} + \underbrace{\psi(|x-y|)}_{\text{local ellipticity}} \right\} \tag{5}$$

- Estimate (5) is an evidence of ergodicity of the underlying process.

  Similar results exist in the probabilistic literature in the form of *contraction estimates for transition probabilities in Wasserstein's metrics* [F.Y. Wang], [Eberle], [Schilling-J.Wang], [Majka]...
- Our proof is entirely analytic and quite elementary: max. principle + doubling variables' method

  We upgrade (especially for nonlocal diffusions) the method developed in [Ishii-Lions '92] for viscosity solutions, extended to nonlocal operators in [Barles-Chasseigne-Imbert '11], [Barles-Chasseigne-Ciomaga-Imbert '13] (see also [Barles-Ley-Topp '17], [Chasseigne-Ley-Nguyen]..)

Decay of weighted seminorms:

$$u(t,x) - u(t,y) \leq e^{-\omega t} \left\{ K \underbrace{\left[\langle x \rangle^k + \langle y \rangle^k\right]}_{\text{Lyapunov}} + \underbrace{\psi(|x-y|)}_{\text{local ellipticity}} \right\} \tag{5}$$

- Estimate (5) is an evidence of ergodicity of the underlying process.

  Similar results exist in the probabilistic literature in the form of *contraction estimates for transition probabilities in Wasserstein's metrics* [F.Y. Wang], [Eberle], [Schilling-J.Wang], [Majka]...

- Our proof is entirely analytic and quite elementary:
  max. principle + doubling variables' method

  We upgrade (especially for nonlocal diffusions) the method developed in [Ishii-Lions '92] for viscosity solutions, extended to nonlocal operators in [Barles-Chasseigne-Imbert '11], [Barles-Chasseigne-Ciomaga-Imbert '13] (see also [Barles-Ley-Topp '17], [Chasseigne-Ley-Nguyen]..)

Key-point: [P.-Priola '12]:

PDE doubling variables methods $\leftrightarrow$ probabilistic coupling methods

Coupling method in probability

[Doeblin '38], [Lindvall, Rogers '86], [Chen-Li '89], [F.Y. Wang '11]...

Given a process X_t starting from $x \in \mathbb{R}^d$, Y_t starting from $y \in \mathbb{R}^d$
$\rightsquigarrow$ look for a new process Z_t in the product space $\mathbb{R}^{2d}$:

(i) the marginal laws of Z_t are the laws of X_t, Y_t respectively

(ii) $Z_t = (X_t, X_t)$ after the first time Z_t hits the diagonal $\Delta := \{x = y\}$.

Coupling method in probability

[Doeblin '38], [Lindvall, Rogers '86], [Chen-Li '89], [F.Y. Wang '11]...

Given a process X_t starting from $x \in \mathbb{R}^d$, Y_t starting from $y \in \mathbb{R}^d$
$\rightsquigarrow$ look for a new process Z_t in the product space $\mathbb{R}^{2d}$:

(i) the marginal laws of Z_t are the laws of X_t, Y_t respectively

(ii) $Z_t = (X_t, X_t)$ after the first time Z_t hits the diagonal $\Delta := \{x = y\}$.

Goal: optimize the estimate

$$u(t,x) - u(t,y) = \mathbb{E}_{Z_t}\left[u_0(x_t) - u_0(y_t)\right] \leq 2\|u_0\|_\infty \mathbb{P}(t < T_c)$$

$\rightsquigarrow$ the Lipschitz estimate is reduced to estimate the hitting time of the diagonal T_c (best over all couplings !)

Coupling method in probability

[Doeblin '38], [Lindvall, Rogers '86], [Chen-Li '89], [F.Y. Wang '11]...

Given a process X_t starting from $x \in \mathbb{R}^d$, Y_t starting from $y \in \mathbb{R}^d$
$\rightsquigarrow$ look for a new process Z_t in the product space $\mathbb{R}^{2d}$:

(i) the marginal laws of Z_t are the laws of X_t, Y_t respectively

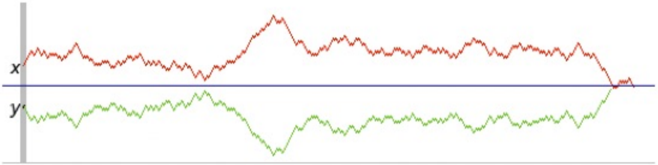
(ii) $Z_t = (X_t, X_t)$ after the first time Z_t hits the diagonal $\Delta := \{x = y\}$.

Goal: optimize the estimate

$$u(t,x) - u(t,y) = \mathbb{E}_{Z_t}[u_0(x_t) - u_0(y_t)] \leq 2\|u_0\|_\infty \mathbb{P}(t < T_c)$$

$\rightsquigarrow$ the Lipschitz estimate is reduced to estimate the hitting time of the diagonal T_c (best over all couplings !)

Ex (coupling by reflection) [from W. Kendall' s course, Warwick '17]



[P-Priola '12] $\rightsquigarrow$ the analytical version:

u, v are sub/super sol. of $\quad \partial_t u = \mathrm{tr}\left(q(x)D^2 u\right) + b(x)\, Du$ in $\mathbb{R}^d$

$\Rightarrow \quad z(x,y) := u(x) - v(y)$ is a subsolution in $\mathbb{R}^{2d}$ *of*

$$\begin{cases} \partial_t z = \mathcal{A}_c(z) & \text{in } \mathbb{R}^{2d} \\ \mathcal{A}_c = \mathrm{tr}\left(q(x)D_x^2 + q(y)D_y^2 + 2c(x,y)D_{xy}^2\right) - b(x)\, D_x - b(y)\, D_y \end{cases}$$

for every choice of the coupling diffusion $c(x,y)$ such that $\mathcal{A}_c$ is elliptic

[P-Priola '12] $\rightsquigarrow$ the analytical version:

u, v are sub/super sol. of $\quad \partial_t u = \mathrm{tr}\left(q(x)D^2u\right) + b(x)\,Du$ in $\mathbb{R}^d$

$\Rightarrow \quad z(x,y) := u(x) - v(y)$ is a subsolution in $\mathbb{R}^{2d}$ *of*

$$\begin{cases} \partial_t z = \mathcal{A}_c(z) & \text{in } \mathbb{R}^{2d} \\ \mathcal{A}_c = \mathrm{tr}\left(q(x)D_x^2 + q(y)D_y^2 + 2c(x,y)D_{xy}^2\right) - b(x)\,D_x - b(y)\,D_y \end{cases}$$

for every choice of the coupling diffusion $c(x,y)$ such that $\mathcal{A}_c$ is elliptic

Roughly speaking, we have

$$u(t,x) - u(t,y) \le \inf_{\mathcal{A}_c} \{\psi(t,x,y), \; : \; \partial_t\psi - \mathcal{A}_c(\psi) \ge 0 \; \}.$$

$\rightsquigarrow$ find best choice of the coupling matrix $c(x,y)$ and supersolution ψ.

[P-Priola '12] $\rightsquigarrow$ the analytical version:

u, v are sub/super sol. of $\quad \partial_t u = \mathrm{tr}\left(q(x)D^2 u\right) + b(x)\,Du$ in $\mathbb{R}^d$

$\Rightarrow \quad z(x,y) := u(x) - v(y)$ is a subsolution in $\mathbb{R}^{2d}$ *of*

$$\begin{cases} \partial_t z = \mathcal{A}_c(z) & \text{in } \mathbb{R}^{2d} \\ \mathcal{A}_c = \mathrm{tr}\left(q(x)D_x^2 + q(y)D_y^2 + 2c(x,y)D_{xy}^2\right) - b(x)\,D_x - b(y)\,D_y \end{cases}$$

for every choice of the coupling diffusion $c(x,y)$ such that $\mathcal{A}_c$ is elliptic

Roughly speaking, we have

$$u(t,x) - u(t,y) \leq \inf_{\mathcal{A}_c} \{\psi(t,x,y), \;:\; \partial_t\psi - \mathcal{A}_c(\psi) \geq 0 \;\}.$$

$\rightsquigarrow$ find best choice of the coupling matrix $c(x,y)$ and supersolution ψ.

Typical ex: $(q(x) = I)$ $\psi = \psi(|x-y|)$, $c(x,y) = Id - 2(\widehat{x-y} \otimes \widehat{x-y})$

[P-Priola '12] $\rightsquigarrow$ the analytical version:

u, v are sub/super sol. of $\quad \partial_t u = \mathrm{tr}\left(q(x) D^2 u\right) + b(x)\, Du$ in $\mathbb{R}^d$

$\Rightarrow \quad z(x,y) := u(x) - v(y)$ is a subsolution in $\mathbb{R}^{2d}$ *of*

$$\begin{cases} \partial_t z = \mathcal{A}_c(z) & \text{in } \mathbb{R}^{2d} \\ \mathcal{A}_c = \mathrm{tr}\left(q(x) D_x^2 + q(y) D_y^2 + 2c(x,y) D_{xy}^2\right) - b(x)\, D_x - b(y)\, D_y \end{cases}$$

for every choice of the coupling diffusion $c(x,y)$ such that $\mathcal{A}_c$ is elliptic

Roughly speaking, we have

$$u(t,x) - u(t,y) \leq \inf_{\mathcal{A}_c} \{\psi(t,x,y), \ : \ \partial_t \psi - \mathcal{A}_c(\psi) \geq 0 \ \}.$$

$\rightsquigarrow$ find best choice of the coupling matrix $c(x,y)$ and supersolution ψ.

Typical ex: $(q(x) = I)$ $\psi = \psi(|x-y|)$, $c(x,y) = Id - 2(\widehat{x-y} \otimes \widehat{x-y})$

• The method is nonlinear: coupling methods are embedded into doubling variables approach for viscosity solutions

For jump processes, one can use a similar idea: double the variables and prescribe the jumps for y to suitably "adapt" to the jumps of x.

For jump processes, one can use a similar idea: double the variables and prescribe the jumps for y to suitably "adapt" to the jumps of x.

Rough sketch of the argument:

- Look at the maximum points of

$$W(t,x,y) := u(t,x) - u(t,y) - K_t \left\{ \underbrace{[\varphi(x) + \varphi(y)]}_{\text{Lyapunov}} + \underbrace{\psi(|x-y|)}_{\text{concave increasing}} \right\}$$

  Claim: no positive maximum can occur.

For jump processes, one can use a similar idea: double the variables and prescribe the jumps for y to suitably "adapt" to the jumps of x.

Rough sketch of the argument:

- Look at the maximum points of

$$W(t,x,y) := u(t,x) - u(t,y) - K_t \left\{ \underbrace{[\varphi(x)+\varphi(y)]}_{\text{Lyapunov}} + \underbrace{\psi(|x-y|)}_{\text{concave increasing}} \right\}$$

Claim: no positive maximum can occur.

- Let (t,x,y) be a max. point $\rightsquigarrow$ $W(t,x,y) \geq W(t,x+z,y+z)$

$$\rightsquigarrow \quad \mathcal{L}[u](x) - \mathcal{L}[u](y) \geq K_t\,(\mathcal{L}[\varphi](x) + \mathcal{L}[\varphi](y))$$

But we also have

$$W(t,x,y) \geq W(t,x+z,y+Az) \qquad \text{for any matrix } A$$

Ex: $A := Id - 2(\widehat{x-y} \otimes \widehat{x-y})$ (reflection of the jumps)

$\rightsquigarrow$ exploits the concavity of ψ for small interactions.

Key-estimate:

Lemma Suppose that

$$\mathcal{L}[u](x) := \int_{\mathbb{R}^d} \{u(x+z) - u(x) - (Du(x)\cdot z)1_{|z|\leq 1}\}\nu(dz)$$

where the Levy measure ν satisfies, in a neighborhood of the origin

$$\exists \lambda > 0 \quad : \qquad \frac{\lambda}{|z|^{d+\alpha}} \leq \frac{d\nu}{dz} \, .$$

If (x, y) is a local maximum point of the function

$$u(x) - u(y) - ([\varphi(x) + \varphi(y)] + \psi(|x - y|))$$

then

$$\mathcal{L}[u](x) - \mathcal{L}[u](y) \geq [\mathcal{L}[\varphi](x) + \mathcal{L}[\varphi](y)] \\ - 4\lambda \int_0^1 (1-s) \int_B \psi''(r + 2s(\widehat{x-y}\cdot z))|z|^2 \frac{dz}{|z|^{d+\sigma}} ds,$$

where $B := \{z \in \mathbb{R}^d \; : \; |z| < (|x - y| \wedge 1)\}$. □

Comments:

- Similar arguments also apply to get regularizing effects.
  Ex ($\alpha > 1$ + strong confinement)

$$\|Du(t)\|_{L^\infty(\langle x\rangle^{-k})} \leq \frac{K\, e^{-\omega t}}{t^{\frac{1}{\alpha}}}\, [u(0)]_{\langle x\rangle^k} \qquad \forall t > 0\,.$$

- The choice of coupling by reflection is just one natural choice for Levy measures $\nu \gtrsim \frac{\lambda}{|z|^{d+\alpha}}$.

Comments:

- Similar arguments also apply to get regularizing effects.
  Ex ($\alpha > 1$ + strong confinement)

$$\|Du(t)\|_{L^\infty(\langle x\rangle^{-k})} \leq \frac{K\, e^{-\omega t}}{t^{\frac{1}{\alpha}}}\, [u(0)]_{\langle x\rangle^k} \qquad \forall t > 0\,.$$

- The choice of coupling by reflection is just one natural choice for Levy measures $\nu \gtrsim \frac{\lambda}{|z|^{d+\alpha}}$.

  Different choices provide extensions to inhomogeneous fractional diffusions.

Comments:

- Similar arguments also apply to get regularizing effects.
  Ex ($\alpha > 1$ + strong confinement)

$$\|Du(t)\|_{L^\infty(\langle x\rangle^{-k})} \leq \frac{K\, e^{-\omega t}}{t^{\frac{1}{\alpha}}} \, [u(0)]_{\langle x\rangle^k} \qquad \forall t > 0\,.$$

- The choice of coupling by reflection is just one natural choice for Levy measures $\nu \gtrsim \frac{\lambda}{|z|^{d+\alpha}}$.

  Different choices provide extensions to inhomogeneous fractional diffusions.

  Key idea: doubling variables allows to embed the coupling ideas in simple analytic version

Comments:

- Similar arguments also apply to get regularizing effects.
  Ex ($\alpha > 1$ + strong confinement)

$$\|Du(t)\|_{L^\infty(\langle x\rangle^{-k})} \leq \frac{K\, e^{-\omega t}}{t^{\frac{1}{\alpha}}}\, [u(0)]_{\langle x\rangle^k} \qquad \forall t > 0\,.$$

- The choice of coupling by reflection is just one natural choice for Levy measures $\nu \gtrsim \frac{\lambda}{|z|^{d+\alpha}}$.

  Different choices provide extensions to inhomogeneous fractional diffusions.

  Key idea: doubling variables allows to embed the coupling ideas in simple analytic version

  More tricky perspective (future goal): how to export same approach to degenerate Kolmogorov operators (Hormander types of diffusions, kinetic models,...)

Conclusions:

- We suggest a new strategy for proving long time decay rates of Fokker-Planck equations (both local and nonlocal diffusions).

  Our method relies on refined estimates on the decay of weighted seminorms for the adjoint problem.

Conclusions:

- We suggest a new strategy for proving long time decay rates of Fokker-Planck equations (both local and nonlocal diffusions).

  Our method relies on refined estimates on the decay of weighted seminorms for the adjoint problem.

- With a unifying approach we recover (and extend) many results obtained with several different methods, including e.g. [Toscani-Villani], [Bakry-Cattiaux-Guillin], [Kavian-Mischler-Ndao] for local operators, or [Tristani], [Gentil-Imbert], [LaFleche] for fractional Laplacians.

Conclusions:

- We suggest a new strategy for proving long time decay rates of Fokker-Planck equations (both local and nonlocal diffusions).

  Our method relies on refined estimates on the decay of weighted seminorms for the adjoint problem.

- With a unifying approach we recover (and extend) many results obtained with several different methods, including e.g. [Toscani-Villani], [Bakry-Cattiaux-Guillin], [Kavian-Mischler-Ndao] for local operators, or [Tristani], [Gentil-Imbert], [LaFleche] for fractional Laplacians.

- Our proof can be thought as a purely analytic version of the probabilistic approach since [Meyn &Tweedie] and more recently [Douc-Fort-Guillin], [Hairer-Mattingly].

Conclusions:

- We suggest a new strategy for proving long time decay rates of Fokker-Planck equations (both local and nonlocal diffusions).

  Our method relies on refined estimates on the decay of weighted seminorms for the adjoint problem.

- With a unifying approach we recover (and extend) many results obtained with several different methods, including e.g. [Toscani-Villani], [Bakry-Cattiaux-Guillin], [Kavian-Mischler-Ndao] for local operators, or [Tristani], [Gentil-Imbert], [LaFleche] for fractional Laplacians.

- Our proof can be thought as a purely analytic version of the probabilistic approach since [Meyn &Tweedie] and more recently [Douc-Fort-Guillin], [Hairer-Mattingly].

  It contains as a by-product a purely analytical proof of many results on ergodicity or time-contraction rates of diffusion semigroups obtained in the probabilistic literature through coupling methods.

Conclusions:

- We suggest a new strategy for proving long time decay rates of Fokker-Planck equations (both local and nonlocal diffusions).
  Our method relies on refined estimates on the decay of weighted seminorms for the adjoint problem.

- With a unifying approach we recover (and extend) many results obtained with several different methods, including e.g. [Toscani-Villani], [Bakry-Cattiaux-Guillin], [Kavian-Mischler-Ndao] for local operators, or [Tristani], [Gentil-Imbert], [LaFleche] for fractional Laplacians.

- Our proof can be thought as a purely analytic version of the probabilistic approach since [Meyn &Tweedie] and more recently [Douc-Fort-Guillin], [Hairer-Mattingly].

  It contains as a by-product a purely analytical proof of many results on ergodicity or time-contraction rates of diffusion semigroups obtained in the probabilistic literature through coupling methods.

- We have succefully applied those results to analyse long-time convergence of mean field games with Levy operators
  [work in progress with O. Ersland & E. Jakobsen]

Thanks for the attention !