Mean value formulas, maximum principle and Harnack inequality for classical solutions to degenerate Kolmogorov equations

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Agenda

- Introduction;
- uniformly elliptic and parabolic setting;
- subelliptic operators on Carnot groups.

(Joint work with Emanuele Malagoli and Diego Pallara).

Uniformly elliptic operators

$$\mathscr{L} u(x) := \operatorname{div}\left(A(x) Du(x)\right) + \langle b(x), Du(x)\rangle$$
$$A(x) = \left(a_{ij}(x)\right)_{i,j=1,\dots,n}$$
$$b(x) = \big(b_1(x), \dots, b_n(x)\big),$$

$$a_{ij}, b_j, \partial_{x_i} a_{ij}, \partial_{x_j} b_j \in C^{\alpha}\big(\Omega\big), \quad i,j = 1, \dots, n.$$

bounded functions, Ω open set of $\mathbb{R}^n$. A symmetric and

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

More differential operators

Uniformly parabolic operators:

$$\mathscr{L} u(x,t) = \operatorname{div}\left(A(x,t) Du(x,t)\right) + \langle b(x,t), Du(x,t)\rangle - \partial_t u(x,t)$$

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$$\mathscr{L} u(x,t) = \operatorname{div}\left(A(x,t)Du(x,t)\right) + \langle b(x,t), Du(x,t)\rangle - \partial_t u(x,t)$$

Degenerate operators on Carnot groups:

$$\mathscr{L} u(x,y,t) = \sum_{j,k=1}^{m} X_j \left(a_{jk}(x,y,t) X_k u(x,y,t)\right) + X_0$$

Examples:

- $X_0 = 0, \quad X_1 = \partial_x + 2y\partial_t, \quad X_2 = \partial_y - 2x\partial_t \ (m=2)$
  $\mathscr{L} = X_1^2 + X_2^2$ (sub-Laplacian on the Heisenberg group)
- $X_1 = \partial_x, \quad X_0 = -\partial_t + x\partial_y \ (m=1)$
  $\mathscr{L} = X_1^2 + X_0$ (degenerate Kolmogorov operator)

Mean value formulas for uniformly elliptic and parabolic equations

The very beginning

Theorem (Cauchy)

Let Ω be an open set of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $z \in \Omega$ and let $\gamma : I \to \Omega$ be a smooth, closed path. Then

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}.$$

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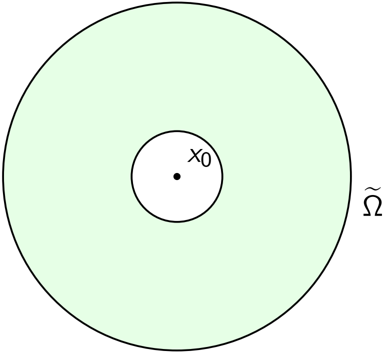
Theorem (Gauss)

Let Ω be an open set of $\mathbb{R}^n$ and let $u \in C^2(\Omega)$. If $\Delta u = 0$ in Ω, and $\overline{B(x,r)} \subset \Omega$, then

$$u(x) = \frac{1}{\mathscr{H}^{n-1}(\partial B(x,r))} \int_{\partial B(x,r)} u(y) d\mathscr{H}^{n-1}(y).$$

About the proof

Choose $\widetilde{\Omega} = B(x_0, r) \backslash B(x_0, \varepsilon)$, $v(x) = c_r - \Gamma(x - x_0)$



About the proof (cont.)

$$0 = \int_{\widetilde{\Omega}} \left(u(x)\Delta v(x) - v(x)\Delta u(x) \right) dx =$$

$$\int_{\partial B(x_0,\varepsilon)} \langle u(x)\nabla v(x) - v(x)\nabla u(x), \nu(x) \rangle d\mathscr{H}^{n-1}(x) -$$

$$\int_{\partial B(x_0,r)} \langle u(x)\nabla v(x) - v(x)\nabla u(x), \nu(x) \rangle d\mathscr{H}^{n-1}(x) \to$$

$$u(x_0) - \int_{\partial B(x_0,r)} \underbrace{-\langle \nabla\Gamma(x - x_0), \nu(x) \rangle}_{(\omega_n r^{n-1})^{-1}} u(x) d\mathscr{H}^{n-1}(x).$$

as we let $\varepsilon \to 0$. Note that: $v = 0$ in $\partial B(x_0, r)$.

The heat equation

Theorem ([Pini] - 1951)

Let Ω be an open set of $\mathbb{R}^2$ and let $u \in C^{2,1}(\Omega)$ satisfying the heat equation $\partial_t u(x,t) = \partial_x^2 u(x,t)$. Then

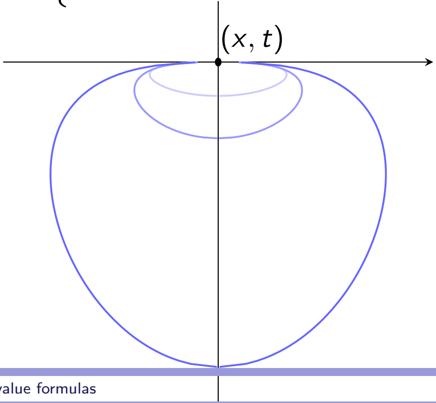
- $$u(x,t) = \frac{1}{\mathscr{H}^1(\partial\Omega_r(x,t))} \int_{\partial\Omega_r(x,t)} K(x,t,y,s) u(y,s) d\mathscr{H}^1(y,s),$$
- $$u(x,t) = \frac{1}{\mu(\Omega_r(x,t))} \int_{\Omega_r(x,t)} \frac{(x-y)^2}{4(t-s)^2} u(y,s) dy\, ds.$$

Here $$\Omega_r(x,t) := \left\{ (y,s) \in \mathbb{R}^2 \mid \Gamma(x,t,y,s) > \frac{1}{r} \right\} \subset \Omega.$$

The set $\Omega_r(x,t)$

$$\partial\Omega_r(x,t) := \left\{ (y,s) \in \mathbb{R}^2 \mid \Gamma(x,t,y,s) = \frac{1}{r} \right\} =$$

$$\left\{ (y,s) \in \mathbb{R}^2 \mid (x-y)^2 = 4(t-s)\left(\log(r) - \frac{1}{2}\log(4\pi(t-s))\right) \right\}$$



Further results

Uniformly parabolic equations

- Montaldo (1955)
- Watson (1973) - several variables
- Kuptsov (1978) - bounded kernels
- Fabes, Garofalo (1987) - smooth coefficients
- Garofalo, Lanconelli (1989)

Degenerate equations

- Citti, Garofalo, Lanconelli (1993) - Sub-Laplacians
- Garofalo, Lanconelli (1990) - Kolmogorov equation
- Lanconelli, P. (1994) - Kolmogorov equation
- P. (1995) - Kolmogorov - smooth coefficients
- Lanconelli, Pascucci (1999) - Hörmander's operators
- Cupini, Lanconelli (2021) - smooth coefficients

Uniformly parabolic operators

$$\mathscr{L} u(x,t) = \operatorname{div}\left(A(x,t)Du(x,t)\right) + \langle b(x,t), Du(x,t)\rangle - \partial_t u(x,t)$$

$$A(x,t) = \left(a_{ij}(x,t)\right)_{i,j=1,\dots,n}$$
$$b(x,t) = \big(b_1(x,t), \dots, b_n(x,t)\big),$$

$$a_{ij}, b_j, \partial_{x_i} a_{ij}, \partial_{x_j} b_j \in C^\alpha\big(\Omega\big), \quad i,j = 1, \dots, n.$$

bounded functions, Ω open set of $\mathbb{R}^{n+1}$. A symmetric and

$$\lambda|\xi|^2 \le \langle A(x,t)\xi, \xi\rangle \le \Lambda|\xi|^2$$

for every $(x,t) \in \Omega$ and $\xi \in \mathbb{R}^n$.

Main results

Theorem 1 [Malagoli, Pallara, P.] (submitted) Let u be a classical solution to $\mathscr{L} u = f$ in Ω. Then, for every $(x_0, t_0) \in \Omega$ and for almost every $r > 0$ such that $\overline{\Omega_r(x_0, t_0)} \subset \Omega$ we have

$$u(x_0, t_0) = \int_{\partial\Omega_r(x_0,t_0)} K(x_0, t_0; x, t) u(x, t)\, d\mathscr{H}^n(x, t) + \int_{\Omega_r(x_0,t_0)} f(x, t) \left(\tfrac{1}{r^n} - \Gamma(x_0, t_0; x, t)\right)\, dx\, dt$$

$$K(x_0, t_0; x, t) = \frac{\langle A(x, t)\nabla_x \Gamma(x_0, t_0; x, t), \nabla_x \Gamma(x_0, t_0; x, t)\rangle}{|\nabla_{(x,t)}\Gamma(x_0, t_0; x, t)|}.$$

Main results

Theorem 1 (cont) Let u be a classical solution to $\mathscr{L} u = f$ in Ω. Then, for every $(x_0, t_0) \in \Omega$ and for every $r > 0$ such that $\Omega_r(x_0, t_0) \subset \Omega$ we have

$$u(x_0, t_0) = \frac{1}{r^n} \int_{\Omega_r(x_0,t_0)} M(x_0, t_0; x, t) u(x, t)\, dx\, dt +$$

$$\frac{n}{r^n} \int_0^r \left(\varrho^{n-1} \int_{\Omega_\varrho(x_0,t_0)} f(x, t) \left(\tfrac{1}{\varrho^n} - \Gamma(x_0, t_0; x, t) \right) dx\, dt \right) d\varrho$$

$$M(x_0, t_0; x, t) = \frac{\langle A(x, t) \nabla_x \Gamma(x_0, t_0; x, t), \nabla_x \Gamma(x_0, t_0; x, t) \rangle}{\Gamma(x_0, t_0; x, t)^2}.$$

About the proof

Consider the function $v(x,t) = \Gamma(x_0, t_0; x, t) - \frac{1}{\varrho^n}$ and apply the divergence theorem to

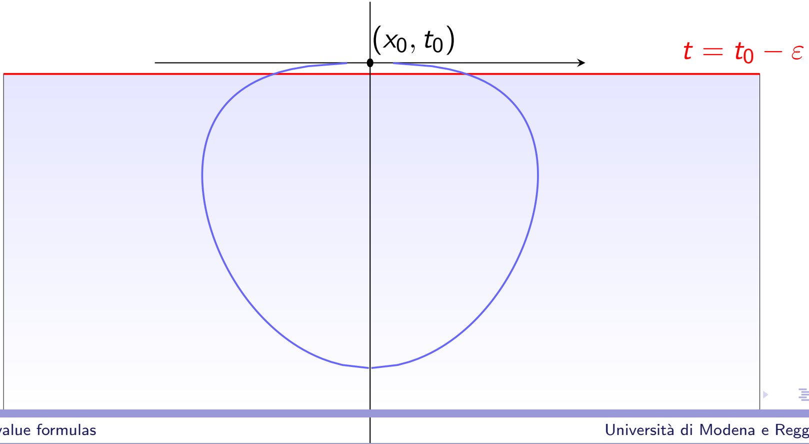
$$\int_\Omega \Big(u(x,t) \underbrace{\mathscr{L}^* v(x,t)}_{0} - v(x,t) \underbrace{\mathscr{L} u(x,t)}_{f(x,t)} \Big) dx\, dt =$$

$$\int_\Omega \Big(\mathrm{div}_x \big(u(x,t) A(x,t) \nabla_x v(x,t) - v(x,t) A(x,t) \nabla_x u(x,t) \big) -$$

$$\mathrm{div}_x \big(u(x,t) v(x,t) b(x,t) \big) + \partial_t \big(u(x,t) v(x,t) \big) \Big) dx\, dt$$

About the proof (cont. 1)

$$\Omega_r^{\varepsilon}(x_0,t_0) := \left\{ (x,t) \in \mathbb{R}^{n+1} \mid \Gamma(x_0,t_0;x,t) > \frac{1}{\varrho^n}, t < t_0 - \varepsilon \right\}$$



Divergence formula

Weak formulation of the divergence theorem (Maggi's book - 2012).

$$F \in C^1(\mathbb{R}^{n+1}), \quad \Omega = \Big\{(x,t) \in \mathbb{R}^{n+1} \mid F(x,t) > c\Big\}$$

$$\Phi \in C^1(\Omega, \mathbb{R}^{n+1}) \cap C\left(\overline{\Omega}, \mathbb{R}^{n+1}\right)$$

If $\mathscr{H}^n\left(\mathrm{Crit}(F) \cap \partial\Omega\right) = 0$, then

$$\int_\Omega \mathrm{div}\ \Phi(x,t) dx\, dt = - \int_{\partial\Omega} \langle \Phi(x,t), \nu(x,t) \rangle d\mathscr{H}^n(x,t)$$

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Partial regularity of $\partial\Omega$

- Dubovickiĭ (1957)
- Bojarski, Hajłasz, and Strzelecki (2005)

Subelliptic operators on Carnot groups

Prototype equation

$$u_t(x,y,t) = \Delta_x u(x,y,t) + \langle x, \nabla_y u(x,y,t)\rangle \qquad (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

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[Kolmogorov](1934) Fundamental solution ($n = 1$) with singularity at $(0,0,0)$

$$\Gamma(x,y,t) = \frac{\sqrt{3}}{2\pi t^2} \exp\left(-\frac{x^2}{t} - 3\frac{xy}{t^2} - 3\frac{y^2}{t^3} \right).$$

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$$\Gamma(x_0,y_0,t_0;x,y,t) = \Gamma(x_0 - x, y_0 - y - x_0(t_0 - t), t_0 - t)$$

Hypoellipticity

Theorem ([Hörmander] - 1967)

Let u be a (distributional) solution to $X_1^2 u + \dots, X_m^2 u + Yu = f$ in $\Omega \subset \mathbb{R}^N \times \mathbb{R}$. If

$$span\Big\{ Y, X_1, \dots, X_m, [X_i, X_j], [X_i, Y], \dots, [X_i, \dots, [X_j, X_l]] \Big\} = \mathbb{R}^{N+1}$$

Then

$$f \in C^\infty(\Omega) \quad \Rightarrow \quad u \in C^\infty(\Omega).$$

Commutators: $[X_i, X_j]f := X_i X_j f - X_j X_i f$

Kolmogorov operator

$$\mathscr{L} := \partial_x^2 + x\partial_y - \partial_t = X^2 + Y$$

- $X = \partial_x, \quad Y = x\partial_y - \partial_t,$

$$Y \sim \begin{pmatrix} 0 \\ x \\ -1 \end{pmatrix} \qquad X \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad [X, Y] = XY - YX \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[X, Y]f := \partial_x(x\partial_y - \partial_t)f - (x\partial_y - \partial_t)\partial_x f = \partial_y f$$

Variable coefficients

$$\mathscr{L} u = \mathrm{div}_x \left(A(x,y,t) D_x u \right) + \langle x, D_y u \rangle - \partial_t u, \quad (x,y,t) \in \mathbb{R}^{2n+1}.$$

Theorem

If $A(x,y,t) := \left(a_{jk}(x,y,t)\right)_{j,k=1,\dots,n}$ is symmetric, uniformly positive, with bounded $C^{1,\alpha}$ coefficients, then a fundamental solution Γ for $\mathscr{L}$ exists, with estimates:

$$c_T^- \Gamma^-(x,y,t;\xi,\eta,\tau) \le \Gamma(x,y,t;\xi,\eta,\tau) \le c_T^+ \Gamma^+(x,y,t;\xi,\eta,\tau)$$

for every $(x,y,t), (\xi,\eta,\tau) \in \mathbb{R}^{2n+1}$ with $0 < t - \tau < T$.

Variable coefficients

$$\mathscr{L} u = \mathrm{div}_x \left(A(x, y, t) D_x u \right) + \langle x, D_y u \rangle - \partial_t u, \quad (x, y, t) \in \mathbb{R}^{2n+1}.$$

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for every $(x, y, t), (\xi, \eta, \tau) \in \mathbb{R}^{2n+1}$ with $0 < t - \tau < T$.

Remark. The fundamental solution doesn't belong to C^1.

Finite perimeter sets

Let E be a measurable subset of $\mathbb{R}^{n+1}$. For any open set $\Omega \subset \mathbb{R}^{n+1}$ the perimeter of E in Ω is

$$P(E, \Omega) := \sup\left\{ \int_E \operatorname{div}(\varphi)\, dz : \varphi \in C_c^1\left(\Omega; \mathbb{R}^{n+1}\right),\ \|\varphi\|_\infty \le 1 \right\}.$$

Note that

$$P(E, \Omega) < +\infty \quad \Leftrightarrow \quad \chi_E \in BV(\Omega)$$

and, in this case, $P(E, \Omega) = |D\chi_E|(\Omega)$.

General formula

If E has finite perimeter in some open set $\Omega \subset \mathbb{R}^{n+1}$, then

$$\int_E \mathrm{div}\ \phi(x,t)dx\,dt = -\int_\Omega \langle \phi(x,t), D\chi_E(x,t)\rangle$$

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Remark If $F \in C^1(\Omega)$ and

$$E_c := \{(x,t) \in \Omega \mid F(x,t) > c\}$$

then, for any compact subset $K \subset\subset \Omega$ and for almost every $c \in \mathbb{R}$ we have $P(E_c, K) < +\infty$.

Reduced boundary

We say that $(x, t) \in \Omega$ belongs to the reduced boundary $\mathcal{F}E$ of E if $|D\chi_E|(B_\varrho(x,t)) > 0$ for every $\varrho > 0$ and the limit

$$\nu_E(x,t) := \lim_{\varrho \to 0} \frac{D\chi_E B_\varrho(x,t)}{|D\chi_E B_\varrho(x,t)|}$$

exists in $\mathbb{R}^{n+1}$ and satisfies $|\nu_E(x,t)| = 1$.

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We have $\mathcal{F}E \subset \partial E$ and

$$\int_E \operatorname{div} \phi(x,t) dx\, dt = - \int_{\mathcal{F}E} \langle \phi(x,t), \nu_E(x,t) \rangle d\mathscr{H}^n(x,t)$$

Some remarks

- This approach doesn't require Dubovickiĭ's theorem,
- it extends to Hörmander's operators on Carnot groups

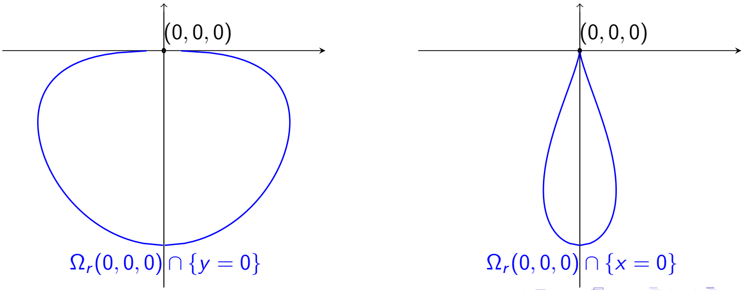
$$\mathscr{L} u(x,t) := \sum_{j,k=1}^{m} X_j \left(a_{jk}(x,t) X_k u(x,t) \right) + X_0 u(x,t) = f(x,t).$$

Remark Classical solutions are not C^1 in the Euclidean sense.

Super level set

$$\mathscr{L} := \partial_x^2 + x\partial_y - \partial_t$$

$$\Gamma(x, y, t) = \frac{\sqrt{3}}{2\pi t^2} \exp\left(-\frac{x^2}{t} - 3\frac{xy}{t^2} - 3\frac{y^2}{t^3}\right).$$



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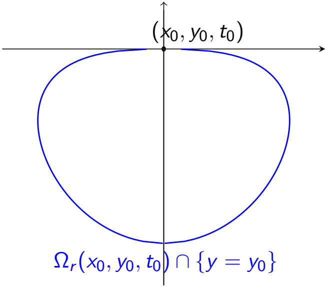
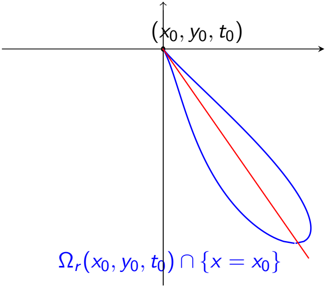
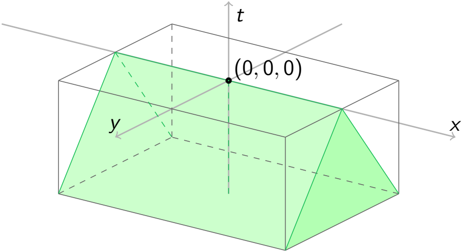




Strong minimum principle

Theorem ([Bony] - 1969, [Amano] - 1979)

Let $u : Q \to \mathbb{R}$ *be a non-negative solution to* $u_{xx} + x u_y = u_t$.
If $u(0,0,0) = 0$, *then* $u \equiv 0$ *in the Propagation set* $\mathcal{A}_{(0,0,0)}$.



Many thanks for your attention!