The space of Hardy-weights for quasilinear equations: Mazya-type characterization and sufficient conditions for existence of minimizers

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The setting

In this talk we consider a nonnegative energy functional

$$Q_{p,A,V}[\varphi] \triangleq \int_\Omega \left(|\nabla\varphi|_A^p + V|\varphi|^p\right) \mathrm{d}x \geq 0 \qquad \forall\phi \in W^{1,p}(\Omega) \cap C_c(\Omega),$$

and its associated Euler-Lagrange equation

$$Q'_{p,A,V}(u) \triangleq \mathrm{div}(|\nabla u|_A^{p-2} A\nabla u) + V|u|^{p-2}u = 0 \qquad \text{in } \Omega,$$

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Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a domain, $1 < p < \infty$, $A \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^{N \times N})$ is a symmetric and locally uniformly positive definite matrix function,

$$|\xi|_A^2 \triangleq \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \qquad x \in \Omega,\ \xi \in \mathbb{R}^N,$$

and V is a real valued potential in a certain local Morrey space $M^q_{\mathrm{loc}}(p; \Omega)$.

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The operator $\Delta_{p,A}[u] \triangleq \mathrm{div}(|\nabla u|_A^{p-2} A\nabla u)$ is called the (p,A)-Laplacian.

Agmon-Allegretto-Piepenbrink-type (AAP)-theorem

Theorem (YP-Psaradakis (2016))

$Q_{p,A,V} \geq 0$ *on* $W^{1,p}(\Omega) \cap C_c(\Omega)$ *iff the equation* $Q'_{p,A,V}(u) = 0$ *in* Ω *admits a positive weak solution (or positive supersolution) in* $W^{1,p}_{\mathrm{loc}}(\Omega)$.

Corollary

$Q_{p,A,V} \geq 0$ *on* $W^{1,p}(\Omega) \cap C_c(\Omega)$ *iff the generalized weak maximum principle holds in any subdomain* $\omega \Subset \Omega$. *That is,*

$$Q'_{p,A,V}(u) \geq 0 \text{ in } \omega, \text{ and } u \geq 0 \text{ on } \partial\omega \quad \Rightarrow \quad u \geq 0 \text{ in } \omega.$$

Picone identity and the simplified energy

Let $\mathbf{u}$ be a positive solution of $Q'_{p,A,V}(u) = 0$ in Ω. Then for all $0 \le \phi \in C_c^\infty(\Omega)$ the following Picone-type identity holds:

$$Q_{p,A,V}(\mathbf{u}\phi) = \int_\Omega \Big[|\phi\nabla\mathbf{u} + \mathbf{u}\nabla\phi|_A^p - \phi^p|\nabla\mathbf{u}|_A^p - p\phi^{p-1}\mathbf{u}|\nabla\mathbf{u}|_A^{p-2}\nabla\mathbf{u}\cdot A\nabla\phi \Big] \mathrm{d}x.$$

Theorem (YP-Tertikas-Tintarev (2008))

Let $\mathbf{u}$ be a positive solution of $Q'_{p,A,V}(u) = 0$ in Ω. Then for all $\mathbf{u}\phi \in W^{1,p}(\Omega) \cap C_c(\Omega)$ we have

$$Q_{p,A,V}(\mathbf{u}\phi) \asymp E_{\mathbf{u}}(\phi) \triangleq \int_\Omega \mathbf{u}^2 |\nabla\phi|_A^2 (\phi|\nabla\mathbf{u}|_A + \mathbf{u}|\nabla\phi|_A)^{p-2} \mathrm{d}x,$$

where the equivalence constant depends only on p. The functional $E_{\mathbf{u}}$ is called the simplified energy of $Q_{p,A,V}$.

$\mathcal{H}_p(\Omega, V)$ the space of Hardy-weights

First aim: characterize the space $\mathcal{H}_p(\Omega, V)$ of all Hardy-weights, i.e., functions $g \in L^1_{\mathrm{loc}}(\Omega)$ such that the following Hardy-type inequality holds:

$$\int_\Omega |g||\phi|^p \, \mathrm{d}x \leq C Q_{p,A,V}(\phi) \qquad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega)$$

for some $C > 0$.

Suppose that $Q_{p,A,V} \geq 0$ in Ω. If $\mathcal{H}_p(\Omega, V) = \{0\}$, then $Q_{p,A,V}$ is said to be critical in Ω, otherwise, $Q_{p,A,V}$ is subcritical in Ω. If $Q_{p,A,V} \not\geq 0$ in Ω, then $Q_{p,A,V}$ is said to be supercritical in Ω.

$Q_{p,A,V}$-capacity

Definition

Let $\mathbf{u}$ be a positive solution of $Q_{p,A,V}(u) = 0$ in Ω. For a compact set $F \Subset \Omega$, the $Q_{p,A,V}$-capacity of F with respect to $(\mathbf{u}, \Omega)$ is defined by

$$\mathrm{Cap}_{\mathbf{u}}(F, \Omega) \triangleq \inf\{Q_{p,A,V}(\phi) \mid \phi \in \mathcal{N}_{F,\mathbf{u}}(\Omega)\},$$

where $\mathcal{N}_{F,\mathbf{u}}(\Omega) \triangleq \{\phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \mid \phi \geq \mathbf{u} \text{ on } F\}$.

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where $\mathcal{N}_{F,\mathbf{u}}(\Omega) \triangleq \{\phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \mid \phi \geq \mathbf{u} \text{ on } F\}$.

We equip $\mathcal{H}_p(\Omega, V)$ with the norm:

$$\|g\|_{\mathcal{H}_p^{\mathbf{u}}(\Omega,V)} \triangleq \sup\left\{\frac{\int_F |g||\mathbf{u}|^p \,\mathrm{d}x}{\mathrm{Cap}_{\mathbf{u}}(F, \Omega)} \mid F \Subset \Omega \text{ compact s.t. } \mathrm{Cap}_{\mathbf{u}}(F, \Omega) \neq 0\right\}.$$

In fact, $\mathcal{H}_p(\Omega, V)$ is a Banach function space. Moreover, in the subcritical case, certain weighted Lebesgue spaces are embedded in $\mathcal{H}_p(\Omega, V)$.

Maz'ya-type characterization of Hardy-weights

$$\|g\|_{\mathcal{H}^{\mathbf{u}}_p(\Omega,V)} = \sup\left\{ \frac{\int_F |g||\mathbf{u}|^p \,\mathrm{d}x}{\mathrm{Cap}_{\mathbf{u}}(F,\Omega)} \mid F \Subset \Omega \text{ compact s.t. } \mathrm{Cap}_{\mathbf{u}}(F,\Omega) \neq 0 \right\}.$$

Theorem (YP - Das (2022))

Let $p \in (1,\infty)$, and $\mathbf{u}$ be a positive solution of $Q_{p,A,V}(u) = 0$ in Ω. Let $g \in L^1_{\mathrm{loc}}(\Omega)$, then $\|g\|_{\mathcal{H}^{\mathbf{u}}_p(\Omega,V)} < \infty$ iff the Hardy-type inequality

$$\int_\Omega |g||\phi|^p \,\mathrm{d}x \leq C Q_{p,A,V}(\phi) \qquad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \qquad \text{(HI)}$$

holds. Moreover, let $\mathcal{B}_g(\Omega, V)$ be the best constant in (HI), then

$$\|g\|_{\mathcal{H}^{\mathbf{u}}_p(\Omega,V)} \leq \mathcal{B}_g(\Omega, V) \leq C_p \|g\|_{\mathcal{H}^{\mathbf{u}}_p(\Omega,V)},$$

where C_p depends only on p. Furthermore, $\|g\|_{\mathcal{B}(\Omega,V)} \triangleq \mathcal{B}_g(\Omega, V)$ is an equivalent norm on $\mathcal{H}_p(\Omega, V)$. In particular, up to the equivalence relation of norms, the norm $\|\cdot\|_{\mathcal{H}^{\mathbf{u}}_p(\Omega,V)}$ is independent of the positive solution $\mathbf{u}$.

Proof of necessity:

Let $g \in L^1_{\mathrm{loc}}(\Omega)$ be a Hardy-weight. Let $F \subset \Omega$ be a compact set. Then for any ψ such that $\psi\mathbf{u} \in W^{1,p}(\Omega) \cap C_c(\Omega)$ with $\psi \geq 1$ on F

$$\int_F |g||\mathbf{u}|^p \,\mathrm{d}x \leq \int_\Omega |g||\psi\mathbf{u}|^p \,\mathrm{d}x \leq \mathcal{B}_g(\Omega, V) Q_{p,A,V}(\psi\mathbf{u}).$$

Taking the infimum over all $\psi\mathbf{u} \in W^{1,p}(\Omega) \cap C_c(\Omega)$ with $\psi \geq 1$ on F, we get

$$\int_F |g||\mathbf{u}|^p \,\mathrm{d}x \leq \mathcal{B}_g(\Omega, V)\mathrm{Cap}_{\mathbf{u}}(F, \Omega) \qquad \text{for all compact sets } F \text{ in } \Omega.$$

Hence, $\|g\|_{\mathcal{H}_p^{\mathbf{u}}(\Omega,V)} \leq \mathcal{B}_g(\Omega, V)$.

Proof of Sufficiency:

Let $\|g\|_{\mathcal{H}_p^{\mathbf{u}}(\Omega,V)} < \infty$ and $0 \leq \psi \in C_c^\infty(\Omega)$. Define

$$\psi_j(x) \triangleq \begin{cases} 0 & \text{if } \psi \leq 2^{j-1}, \\ \left[\frac{\psi}{2^{j-1}} - 1\right]^\alpha & \text{if } 2^{j-1} \leq \psi \leq 2^j, \\ 1 & \text{if } 2^j \leq \psi, \end{cases} \qquad -\infty < j < \infty,$$

where $\alpha = 1$ if $p \geq 2$, and $\alpha = 2/p$ if $p < 2$.

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where $\alpha = 1$ if $p \ge 2$, and $\alpha = 2/p$ if $p < 2$. Since

$$\int_F |g||\mathbf{u}|^p \,\mathrm{d}x \le \|g\|_{\mathcal{H}_p^{\mathbf{u}}(\Omega,V)} \mathrm{Cap}_{\mathbf{u}}(F,\Omega), \ \forall F \Subset \Omega \text{ compact}, \ \mathrm{Cap}_{\mathbf{u}}(F,\Omega) \ne 0.$$

$$\Longrightarrow \int_{\{\psi \ge 2^j\}} |g||\mathbf{u}|^p \,\mathrm{d}x \le \|g\|_{\mathcal{H}_p^{\mathbf{u}}(\Omega,V)} Q_{p,A,V}(\mathbf{u}\psi_j) \quad \psi_j = 1 \text{ on } \psi \ge 2^j.$$

Using the co-area formula, replacing $Q_{p,A,V}$ with the simplified energy, estimating ψ_j and $\nabla\psi_j$, and finally summing up, one obtains

$$\int_\Omega |g||\mathbf{u}\psi|^p \,\mathrm{d}x \le C_p \|g\|_{\mathcal{H}_p^{\mathbf{u}}(\Omega,V)} Q_{p,A,V}(\mathbf{u}\psi).$$

Definition (Beppo Levi space)

The generalized Beppo Levi space $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$ is the completion of $W^{1,p}(\Omega) \cap C_c(\Omega)$ with respect to the norm

$$\|\phi\|_{\mathcal{D}^{1,p}_{A,V^+}(\Omega)} \triangleq \left[\||\nabla\phi|_A\|^p_{L^p(\Omega)} + \|\phi\|^p_{L^p(\Omega, V^+ \,\mathrm{d}x)} \right]^{1/p} .$$

For $g \in \mathcal{H}_p(\Omega, V)$, consider the generalized principal eigenvalue

$$\mathbb{S}_g(\Omega, V) \triangleq \inf\{Q_{p,A,V}(\phi) \mid \phi \in W^{1,p}(\Omega) \cap C_c(\Omega)\,, \int_\Omega |g||\phi|^p \,\mathrm{d}x = 1\}.$$

In fact,

$$\mathbb{S}_g(\Omega, V) = \inf\{Q_{p,A,V}(\phi) \mid \phi \in \mathcal{D}^{1,p}_{A,V^+}(\Omega)\,, \int_\Omega |g||\phi|^p \,\mathrm{d}x = 1\}.$$

We say that the best constant $\mathcal{B}_g(\Omega, V)$ is attained if $\mathbb{S}_g(\Omega, V) = (\mathcal{B}_g(\Omega, V))^{-1}$ is attained in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.

Sufficient condition for attainment of the best constant

Let $\quad \mathcal{H}_{p,0}(\Omega, V) \triangleq \overline{\mathcal{H}_p(\Omega, V) \cap L^\infty_c(\Omega)}^{\|\cdot\|_{\mathcal{H}_p(\Omega,V)}}$.

Theorem (For the case $V = V^+$)

If $g \in \mathcal{H}_{p,0}(\Omega, V^+)$, then the functional

$$T_g(\phi) \triangleq \int_\Omega |g||\phi|^p \,\mathrm{d}x$$

is compact on $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$. In particular, $\mathcal{B}_g(\Omega, V^+)$ is attained in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.

Since $\mathcal{H}_{p,0}(\Omega, V) \subset \mathcal{H}_{p,0}(\Omega, V^+)$, it follows that if $g \in \mathcal{H}_{p,0}(\Omega, V)$, then T_g is compact in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.

Attainment of the best constant I (spectral gap condition)

Theorem

Assume that $A \in C^{\gamma}$, $g \in \mathcal{H}_p(\Omega, V) \cap \mathcal{M}^q_{\rm loc}(p;\Omega)$, *and*

$$\mathbb{S}_g(\Omega) < \mathbb{S}_g^{\overline{\infty}}(\Omega) \triangleq \sup\{K \Subset \Omega \mid \mathbb{S}_g(\Omega \setminus K)\}.$$

Assume that $\int_{\Omega\setminus K_1} V^- G^p \,\mathrm{d}x < \infty$, *where* G *is a positive solution of the equation* $Q'_{p,A,V-\mathbb{S}_g(\Omega)|g|}[u] = 0$ *in* $\Omega \setminus K$ *of minimal growth in a neighborhood of infinity in* Ω, *where* $K \Subset K_1 \Subset \Omega$.
Then, $\mathcal{B}_g(\Omega)$ *is attained in* $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.

Proof's outline: If $\mathbb{S}_g(\Omega) < \mathbb{S}_g^{\overline{\infty}}(\Omega)$, then $Q_{p,A,V-\mathbb{S}_g(\Omega)|g|}$ is critical in Ω, with a ground state Φ and a null-sequence $0 \le \phi_n \le \Phi$. It follows that $\int_\Omega (V^- + |g|)\Phi^p \,\mathrm{d}x < \infty$, $\hat{\phi}_n \rightharpoonup \Phi$ in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$, and we may pass to the limit in the identity:

$$\int_\Omega |\nabla \phi_n|_A^p \mathrm{d}x + \int_\Omega V_+ |\phi_n|_A^p \mathrm{d}x = Q_{p,A,V-\mathbb{S}_g(\Omega)|g|}(\phi_n) + \int_\Omega V_- \phi_n^p \mathrm{d}x + \mathbb{S}_g(\Omega)\int_\Omega |g|\phi_n^p \mathrm{d}x.$$

Attainment of the best constant II (spectral gap condition)

For $x \in \overline{\Omega}$ and $g \in \mathcal{H}_p(\Omega, V)$, define the Hardy constant of g at x by

$$\mathbb{S}_g(x,\Omega) \triangleq \liminf_{r\to 0}\{Q_{p,A,V}(\phi) \mid \phi \in \mathcal{D}^{1,p}_{A,V^+}(\Omega\cap B_r(x)), \int_{\Omega\cap B_r(x)}|g||\phi|^p \mathrm{d}x = 1\},$$

and let $\Sigma_g \triangleq \{x \in \overline{\Omega} \mid \mathbb{S}_g(x,\Omega) < \infty\}$, and $\mathbb{S}^*_g(\Omega) \triangleq \inf_{x\in\overline{\Omega}} \mathbb{S}_g(x,\Omega)$.

Theorem

Let Ω be a bounded domain, $V \in \mathcal{M}^q_{\mathrm{loc}}(p;\Omega)$ s.t. $V^- \in \mathcal{H}_{p,0}(\Omega, V^+)$.
Suppose that $g \in \mathcal{H}_p(\Omega, V)$ s.t. $\left|\overline{\Sigma_g}\right| = 0$.
If $\mathbb{S}_g(\Omega) < \mathbb{S}^*_g(\Omega)$, then $\mathcal{B}_g(\Omega)$ is attained in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.

The proof relies on concentration compactness arguments.

Thank you for your attention!