Regularity for Supersolutions to Fully Nonlinear PDEs Under Convexity Assumptions

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May 2022

Mostly Maximum Principle - (Cortona)

Joint work with Alessio Figalli (ETH) and J. Ederson M. Braga (UFC), Edgard Pimentel (University of Coimbra)

Outline

References

- Braga, J. Ederson M.; Moreira, Diego *Inhomogeneous Hopf-Oleinik lemma and regularity of semiconvex supersolutions via new barriers for the Pucci extremal operators.* Adv. Math. 334 (2018), 184-242.

- Braga, J. Ederson M.; Figalli, Alessio; Moreira, Diego *Optimal regularity for the convex envelope and semiconvex functions related to supersolutions of fully nonlinear elliptic equations.* Comm. Math. Phys. 367 (2019), no. 1, 1-32.

Analytic control above and below

$$u \in C^0(B_1), \quad -C \leq \Delta u \leq C \ \left(|\Delta u| \leq C\right)$$

Then,

$$u \in C^{1,\alpha}(B_1) \cap W^{2,p}_{loc}(B_1) \quad \forall \alpha \in (0,1), \ \forall p \in (1,\infty)$$

with

$$||u||_{C^{1,\alpha}(B_{\frac{1}{2}})} \leq C(n)\left(||u||_{L^\infty(B_1)} + C\right)$$

$$||u||_{W^{2,p}(B_{\frac{1}{2}})} \leq C(n)\left(||u||_{L^\infty(B_1)} + C\right)$$

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Analytic from above and geometric from below ?

Analytic control from above

supersolution $(\Delta u \leq f$ in B_1 with $f \in L^q(B_1))$

Geometric control from below

Some kind of convexity of u

Questions:

$a)$ Can we prove regularity for u (weak aolution) ?

$b)$ Can we obtain optimal regularity of u based on regularity of RHS and (modulus) of convexity from below ?

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Classical Solutions: Example I

Coming back to the question

Supersolutions (analytic control from above)

$$u \in C^2(B_1)$$

$$\Delta u \leq 0 \quad \text{in} \quad B_1$$

Convexity (Geometric control from below)

$$D^2 u \geq 0 \quad \text{in} \quad B_1$$

(Supporting plane from below everywhere)

$$0 \leq ||D^2 u(x)|| \leq \Delta u \leq 0 \text{ in } B_1 \Longrightarrow D^2 u \equiv 0 \text{ in } B_1$$

Then, u is affine

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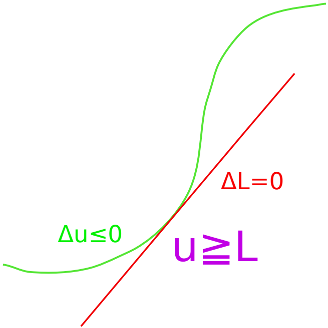
Different Perspective: Maximum Principle Argument

u is affine by a Maximum Principle Argument



Maximum Principle ⟹ u=L

Classical Solutions: Example II

Control (Ana) Above $\&$ (Geo) Below: Example II

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Question: Are there apriori estimates for u ?

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Classical Solutions: Example III

Control Above $\&$ Below: Case III

Supersolutions (analytic control from above)

$$u \in C^2(B_1)$$

$$Lu(x) := tr(A(x)D^2u(x)), \quad \lambda \cdot Id \leq A(x) \leq \Lambda \cdot Id, \; \forall x \in B_1$$

$$Lu \leq C \quad \text{in} \quad B_1$$

Convexity (geometric control from below)

$$D^2u \geq 0 \quad \text{in} \quad B_1$$

(tangent plane from below everywhere)

Question: Are there apriori estimates for u ?

How does the info from equation play out ?

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Extremal Pucci Operators

Pucci Operators - Fully Nonlinear Operators - I

$$0 < \lambda < \Lambda, \quad M \in \mathcal{S}^{n\times n}$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \lambda \cdot Tr(M^+) - \Lambda \cdot Tr(M^-)$$
$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \Lambda \cdot Tr(M^+) - \lambda \cdot Tr(M^-)$$

Multiples of Trace Operator for Nonnegative Matrices

$$M \geq 0 \Longrightarrow \mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \cdot Tr(M) \;\&\; \mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \cdot Tr(M)$$

(Super-additivity of $\mathcal{M}^-_{\lambda,\Lambda}$) $\quad \mathcal{M}^-_{\lambda,\Lambda}(M+N) \geq \mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^-_{\lambda,\Lambda}(N)$

(Sub-additivity of $\mathcal{M}^+_{\lambda,\Lambda}$) $\quad \mathcal{M}^+_{\lambda,\Lambda}(M+N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N)$

Pucci Operators - Fully Nonlinear Operators - II

Homogeneity

$$\lambda > 0 \Longrightarrow \mathcal{M}^{\pm}(\lambda \cdot M) = \lambda \cdot \mathcal{M}^{\pm}(M)$$

$$\lambda < 0 \Longrightarrow \mathcal{M}^{\pm}(\lambda \cdot M) = \lambda \cdot \mathcal{M}^{\mp}(M)$$

Envelope of Linear Operators

$$A \in \mathcal{S}^{n \times n}, \; spec(A) = \sigma(A) = \Big\{\mu; \; \mu \text{ is an eigenvalue of } A\Big\}$$

$$\mathcal{A}_{\lambda,\Lambda} := \Big\{A \in \mathcal{S}^{n \times n}; \; \mu \in \sigma(A) \Rightarrow \mu \in [\lambda, \Lambda]\Big\}$$

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} Trace(AM), \quad \mathcal{M}^{+}_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} Trace(AM)$$

Envelope for Linear Equations

$$u \in C^2(B_1), \quad (UE)\ \ A(x) \in \mathcal{A}_{\lambda,\Lambda},\ \forall x \in B_1.$$

$$Lu(x) := Tr(A(x)D^2u) = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j}(x) \qquad (\star)$$

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u(x)) \leq Lu(x) \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2u(x))$$

$\forall L\ \ (UE)$ operator as in $(\star)$

$Lu \leq f$ in $B_1 \Longrightarrow \mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f$ in B_1

$Lu \geq f$ in $B_1 \Longrightarrow \mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq f$ in B_1

(Equations in blue are Fully Nonlinear Elliptic PDEs)

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Supersolutions (analytic control from above)

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$$Lu \leq C \quad \text{in} \quad B_1$$

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(tangent plane from below everywhere)

Question: Are there apriori estimates for u ?

$$0 \leq \lambda \cdot ||D^2u(x)|| \leq \lambda \cdot Trace(D^2u) \leq \mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq Lu \leq C$$

$$||D^2u(x)|| \leq \lambda^{-1} \cdot C \quad \forall x \in B_1 \quad (u \in C^{1,1} \text{ with estimates})$$

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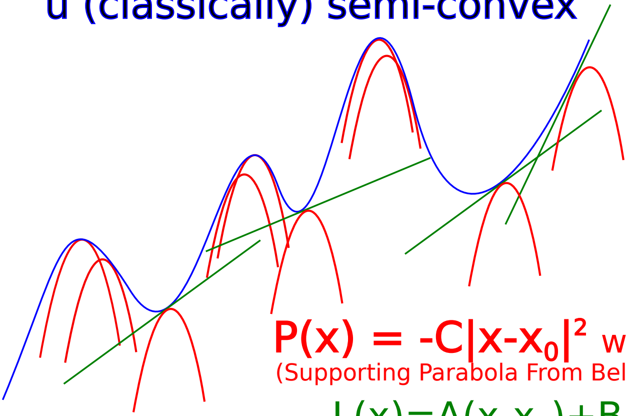
Replacing Convexity by (Classical) Semi-Convexity

Geometric Meaning of (Classical) Semi-Convexity

u (classically) semi-convex



$P(x) = -C|x-x_0|^2$ with $C>0$

(Supporting Parabola From Below)

$L(x)=A(x-x_0)+B$

(Supporting Plane From Below)

Linear Modulus of Semiconvexity

$$\Delta_h^2 u(x) := \frac{u(x+h) + u(x-h) - 2u(x)}{|h|^2}$$

(Second Order Differential Quotient)

u is semiconvex with constant $C > 0 \iff \Delta_h^2 u(x) \geq -C$

Equivalences $(u \in C^0)$

$i)$ $u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y) + \frac{C}{2}|x-y|^2$

$ii)$ $u + \frac{C}{2}|x - x_0|^2$ is convex $(\forall\, x_0)$;

$iii)$ $D^2 u \geq -C \cdot I_n$ in the sense of distributions;

$iv)$ $D^2 u \geq -C \cdot I_n$ in the viscosity sense;

$v)$ u has a concave paraboloid of "opening C" touching from below

$iii)$ & $iv)$ are PDE characterization of semi-convexity

Classical Solutions: Example IV

Control Above & Below: Case IV

$$u \in C^2(B_1), \quad C > 0, \quad f \in L^q(B_1)$$

(analytic control) $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f$ in B_1,

(geometric control) $D^2u \geq -4C \cdot Id$ in B_1

(Parabola from below everywhere)

Question: Are there apriori estimates for u ?

Answer: $W^{2,q}$ Apriori Estimates

Even more: $C^{1,1-\frac{n}{q}}$ Apriori Estimates for $q > n$

(By Morrey-Sobolev Embedding Thm)

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$$D^2u \geq -4C \cdot Id, \quad P(x) := 2C|x|^2,$$

$$v(x) := u(x) + P(x), \quad v \text{ is convex}$$

$$\begin{aligned}
f(x) &\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2u(x)) \\
&\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2v(x) - D^2P(x)) \\
&\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2v(x)) + \mathcal{M}^-_{\lambda,\Lambda}(-D^2P(x)) \\
&\geq \mathcal{M}^-_{\lambda,\Lambda}(D^2v(x)) - \mathcal{M}^+_{\lambda,\Lambda}(D^2P(x)) \\
&\geq \lambda \cdot Tr(D^2v(x)) - 4nC\Lambda \quad (\text{since } v \text{ is convex}) \\
&\geq \lambda \cdot v_{ee}(x) - 4nC\Lambda \\
&= \lambda \cdot (u_{ee}(x) + 4C) - 4nC\Lambda \\
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$$-4C \leq u_{ee}(x) \leq \lambda^{-1}(f(x) + 4nC\Lambda) \quad \forall x \in B_1.$$

This implies

$$||u||_{W^{2,q}(B_{1/2})} \leq D \cdot \Big(C + ||u||_{L^q(B_1)} + ||f||_{L^q(B_1)}\Big).$$

Moreover, if $q > n$ by Sobolev-Morrey Embedding Theorem

$$||u||_{C^{1,1-n/q}(B_{1/2})} \leq \overline{D}\Big(C + ||u||_{L^\infty(B_1)} + ||f||_{L^q(B_1)}\Big)$$

What happens with more complicated notions of convexity ?

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What If a More Complicated Geometry from below?

Questions

- What happens under more complex notion of convexity ?
- Is it possible to prove regularity for weak solutions ?
- Regularity + Apriori Estimates ?

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General Concept of Semi-Convexity

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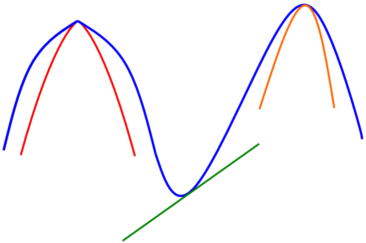

$P(x) = -C|x-x_0|^{1+\alpha}$ with $C>0$

$C^{1,\alpha}$ supporting surface from below

$P(x) = -|x-x_0|\,\omega(|x-x_0|)$

$C^{1,\omega}$ supporting surface from below

$0 \leqq \omega(t) \to 0$ as $t \to 0^+$

Elements of Convex Analysis I: Semi-Convexity

$u : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^n$ bounded and convex

$\omega : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+,$ nondecreasing, upper-semicontinuous, $\omega(0) = 0$

u is $\omega -$ semiconvex iff $\forall x, y \in \Omega$

$$u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y) + \underbrace{\lambda(1-\lambda)|x-y|\omega(|x-y|)}_{C^{1,\omega}\text{(correction)}}$$

$\omega(t) = \frac{Ct}{2}$ we say that u is $C -$ semiconvex

u is $C -$ semiconvex $\Longleftrightarrow D^2 u \geq -C \cdot I_n$ (PDE characterization)

Elements of Convex Analysis II: Semi-Convexity

$\omega-$Normal Mapping

$$\partial_w u(x) := \left\{ \xi \in \mathbb{R}^n ; u(y) \geq u(x) + \xi \cdot (y-x) - |y-x|\omega(|y-x|) \quad \forall y \in \Omega \right\}.$$

$$\partial_\omega u : \Omega \to \mathcal{P}(\mathbb{R}^n), \quad x \mapsto \partial_\omega u(x)$$

$$\partial_w u(x) \neq \varnothing \iff \text{u is } \omega - \text{semiconvex}$$

Proposition 2.1.2 *Let $u \in C^1(A)$, with A open. Then both u and $-u$ are locally semiconcave in A with modulus equal to the modulus of continuity of Du.*

(P. Cannarsa & C. Sinestrari Book)

CKNS Theorem

CKNS apriori estimate

Theorem (Caffarelli, Kohn, Nirenberg, Spruck, CPAM, 1985)

Let $u \in C^2(B_r)$ *be such that*

$i)$ $Lu = Tr(A(x)D^2u) \leq C$ *in* B_r with $\lambda Id \leq A(x) \leq \Lambda Id$

$ii)$ $||u||_{C^1(B_r)} \leq C$;

$iii)$ u *is* $\omega-$*semiconvex in* B_r *where* $\omega(t) = Ct^\alpha$ *for some* $\alpha \in (0,1]$.

Then, there exists $\overline{C} = \overline{C}(n, \lambda, \Lambda, C, \alpha) > 0$ *so that*

$$|\nabla u(x) - \nabla u(y)| \leq \frac{\overline{C}}{1 + |\log|x-y||} \quad \forall x, y \in B_{r/2}. \tag{1}$$

Analytic (above) + Geometric (below) controls

render an estimate on the MC of the Gradient

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BMF Result

Theorem (BFM, 2019, CMP)

Let φ be a bounded and $\omega-$semiconvex viscosity solution to

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) \leq f \;\text{ in }\; B_1$$

Assume that $f \in L^q(B_r)$ with $q \geq n$ and thus $\tau := 1 - n/q \geq 0$. Set

$$\|f\|_{L^n(B_\rho(x_0))} \leq \vartheta(\rho) \tag{2}$$

$$\forall x_0 \in \overline{B}_1, \quad 0 < \rho < 1 - |x_0|.$$

$$\Upsilon(s) := \begin{cases} \omega(4s) + s^{\tau} & \text{if } \; q > n, \\ \\ \omega(4s) + \vartheta(4s) & \text{if } n = q, \text{ with } \vartheta \text{ as in } (2) \text{ above.} \end{cases}$$

Then, $\varphi \in C^{1,\Upsilon}(B_{1/64})$ with precise estimates in $[\nabla\varphi]_{C^{0,\Upsilon}(B_{r/64})}$

Estimates on $[\nabla\varphi]_{C^{0,\Upsilon}(B_{r/64})}$

$$q > n,$$

$$[\nabla\varphi]_{C^{0,\Upsilon}(B_{r/64})} \leq C\left(1 + \|f\|_{L^q(B_r)}\right)$$

$$q = n,$$

$$[\nabla\varphi]_{C^{0,\Upsilon}(B_{r/64})} \leq C\left(1 + \frac{\|\varphi\|_{L^\infty(B_r)}}{r} + \omega(r)\right).$$

Comparing with CKNS result

(RHS Bdd $\Longrightarrow q = \infty \Longrightarrow \tau = 1$)

$$|\nabla\varphi(x) - \nabla\varphi(y)| \leq C\left(|x-y|^\alpha + |x-y|\right) \leq C|x-y|^\alpha$$

(CKNS) $\quad |\nabla\varphi(x) - \nabla\varphi(y)| \leq \dfrac{\overline{C}}{1 + |\log|x-y||}$

$C|x-y|^\alpha \leq \dfrac{\overline{C}}{1 + |\log|x-y||}$ for $|x-y| << 1$

Application of Regularity Theorem for Supersolutions

$$u \in W^{2,n}_{loc}(B_1) \Longrightarrow u \in C^{\alpha}_{loc}(B_1) \quad \forall \alpha \in (0,1)$$

$$u \quad \text{is} \quad \omega - \text{semiconvex} \Longrightarrow u \in C^{0,1}_{loc}(B_1)$$

$$u \in W^{2,n}_{loc}(B_1) \quad \text{and} \quad \omega - \text{semiconvex} \Longrightarrow u \in C^{1}(B_1)$$

Proof: Set $f := \Delta u \in L^n(B_1)$. Then, u is a L^n-strong solution to

$$\Delta u \leq f \ \text{ in } \ B_1.$$

In particular, u is a L^n-viscosity solution to $\Delta u \leq f$ in B_1.

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Estimates Semiconvex Functions

Theorem ((BMF) - Estimates for $\omega-$semiconvex functions)

Let $u \in L^1(B_r)$ be a $\omega-$semiconvex function and $p \in (0, \infty)$.

(a) $u \in C^{0,1}_{loc}(B_r)$;

(b) *There exists $C_1 = C_1(n,p) > 0$ such that*

$$\sup_{B_{r/2}} |u| \leq C_1 \left[\left(\fint_{B_r} |u|^p dx \right)^{1/p} + r\omega(r) \right].$$

(c) *For some $C_3 = C_3(n,p) > 0$ we have*

$$ess \sup_{B_{r/2}} |\nabla u| \leq \frac{C_3}{r} \left[\left(\fint_{B_r} |u|^p dx \right)^{1/p} + r\omega(r) \right]. \qquad (3)$$

Ideas of the Proof

Harnack Approach Flipping the MC Above & Below

Harnack: control by below implies by above

$$u \in C^0(B_1),\ \mathcal{M}^-_{\lambda,\Lambda}(D^2u) \le 0 \le \mathcal{M}^+_{\lambda,\Lambda}(D^2u) \quad \text{in } B_1$$

$u - l$ satisfies Harnack inequality whenever $u - l \ge 0$ in B_1

$\forall l$ affine function (Converse is also true (Caffarelli (1999))

Let l affine function so that $u(0) = l(0)$

Assume that u separates from l by below by (rate) $\omega(r) \ge 0$.

$$\inf_{B_r}(u - l) \ge -\omega(r) \quad \forall r \in (0,1).$$

Setting $v_r(x) := u(x) - l(x) + \omega(r)$ for $x \in B_1$. Then,

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Idea:

“Harnack type argument" reproduces above (with some correction) the $C^{1,\omega}$ regularity existing below (that comes from the semi-convexity) to above

Difficulties to implement in our scenario

- Only Half Harnack is available (Weak Harnack Inequality);
- For class $\overline{S}(\gamma; f)$ equations perceive linear functions;

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- Only Half Harnack is available (Weak Harnack Inequality);
- For class $\overline{S}(\gamma; f)$ equations perceive linear functions;

Key Idea To The Proof of Theorems

Assume that

- φ is $\omega-$semiconvex
- $0 \in \partial_\omega \varphi(0)$
- $\mathcal{M}^-_{\lambda,\Lambda}(D^2\bar{\varphi}) \le f$ in B_r.

Then,

$$\varphi(x) \ge \varphi(0) - |x|\omega(|x|) \;\; \forall x \in B_r$$

This implies,

- $\bar{\varphi} := \varphi - \varphi(0) + r\omega(r) \ge 0 \quad$ in B_r
- $\mathcal{M}^-_{\lambda,\Lambda}(D^2\varphi) \le f$ in B_r
- $\bar{\varphi}$ is $\omega -$ semi-convex

Key Idea To The Proof of Theorems

This implies,

$$\left(\fint_{B_{r/2}} \bar{\varphi}^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} \leq C\left(\inf_{B_r} \bar{\varphi} + r^{1+\alpha}||f||_{L^q(B_r)}\right)$$

$$\leq C\Big(r\omega(r) + r^{1+\alpha}||f||_{L^q(B_r)}\Big), \quad \alpha := 1 - n/q$$

$$||\bar{\varphi}||_{L^\infty(B_{r/4})} \leq D \cdot \left(\fint_{B_{r/2}} \varphi^{\varepsilon_0} dx\right)^{\frac{1}{\varepsilon_0}} + D \cdot r\omega(r) = C\vartheta(r)$$

$$||\bar{\varphi}||_{L^\infty(B_{r/4})} \leq \overline{D}\Big(r \cdot \omega(r) + r^{1+\alpha}||f||_{L^q(B_r)}\Big) =: \overline{D}r \cdot \vartheta(r)$$

$$||\varphi||_{L^\infty(B_{r/4})} \leq \overline{D}\Big(r \cdot \omega(r) + r^{1+\alpha}||f||_{L^q(B_r)}\Big) = (\overline{D} + 1) \cdot r \cdot \vartheta(r)$$

$$\vartheta(r) := \omega(r) + r^{\alpha}||f||_{L^q(B_r)}$$

New Developments (Work with E. Pimentel)

$$(WH)\ \left(\fint_{B_{\rho/2}(x_0)} u^{\varepsilon}\mathrm{d}x\right)^{\frac{1}{\varepsilon}} \le C_{\mathrm{WH}}\left(\inf_{B_{\rho/2}(x_0)} u + \rho^{R}\,\|f\|_{L^q(B_\rho(x_0))}\right)$$

for every $\rho > 0$ and $x_0 \in B_1$ such that $B_\rho(x_0) \subset B_1$

$$(L^\infty - L^\varepsilon)\ \|u\|_{L^\infty(B_{\rho/2}(x_0))} \le C_{\varepsilon,\infty}\left[\left(\fint_{B_\rho(x_0)} u^{\varepsilon}\mathrm{d}x\right)^{\frac{1}{\varepsilon}} + \sigma(\rho)\right]$$

for some $\varepsilon > 0$, and every $\rho > 0$ and $x_0 \in B_1$ such that $B_\rho(x_0) \subset B_1$.

New developments & Results (with E. Pimentel)

- (C1) $u + c$ $(c - u)$ satisfies (WH) and $(L^\infty - L^\varepsilon)$, $\forall c$ constant
- (C2) $u + l$ $(l - u)$ satisfies (WH) and $(L^\infty - L^\varepsilon)$, $\forall l$ affine

  - Weak Harnack Inequality

Nonnegative Supersolutions to Linear + Nonlinear PDEs, Super Q-minimizers, De Giorgi Class DG_p^-

- $L^\infty - L^\varepsilon$ type estimates

Subsolutions to Linear + Nonlinear PDEs, Sub Q-minimizers, De Giorgi Class DG_p^+, $\omega-$semiconvexity, C^α, $C^{1,\alpha}$ from below

- Results: Flipping C^α, $C^{1,\alpha}$ regularities from below + Sobolev Regularity (Besov Spaces) of Supersolutions and Convex Envelope (of supersolutions) (Like L. Caffarelli's Result)

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Nonnegative Supersolutions to Linear + Nonlinear PDEs, Super Q-minimizers, De Giorgi Class DG_p^-

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Thank You Very Much !

LINEAR ALGEBRA REMARKS

Linear Algebra Remarks (I)

$$M = (m_{ij})_{i,j\in\{1,\cdots,n\}} \in \mathcal{S}^{n\times n}$$

$$||M||_1 = \sum_{i,j=1}^{n} |m_{ij}|, \quad ||M||_{spec} = max\{|\lambda|, \ \lambda \in \sigma(M)\}$$

Spectral Theorem

$$\Downarrow$$

$\exists \mathcal{B} = \{e_i\}_{i=1}^n$ orthonormal basis in $\mathbb{R}^n$ with $Me_i = \lambda_i \cdot e_i$

$$\langle Mv, v\rangle = \sum_{i,j=1}^{n} v_i v_j \langle Me_i, e_j\rangle = \sum_{i,j=1}^{n} \lambda_i v_i v_j \delta_{ij} = \sum_{i=1}^{n} \lambda_i v_i^2$$

$$\max_{v\in\mathbb{S}^{n-1}} |\langle Mv, v\rangle| = ||M||_{spec}$$

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$||M||_1 \leq C(n) \cdot ||M||_{spec}$ (Finite Dimensional VS)

$$M \geq 0 \Longrightarrow |m_{ij}| \leq ||M||_1 \leq C(n)||M||_{spec} \leq C||M||_{spec} \leq C \cdot Tr(M)$$

u is convex $\Leftrightarrow D^2u(x) \geq 0 \Longrightarrow |u_{x_ix_j}| \leq C||D^2u(x)|| \leq C \cdot \Delta u(x)$

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Extremal Pucci Operators

Pucci Operators - Fully Nonlinear Equations - I

$$0 < \lambda < \Lambda, \quad M \in \mathcal{S}^{n\times n}$$

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) := \lambda \cdot Tr(M^{+}) - \Lambda \cdot Tr(M^{-})$$

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) := \Lambda \cdot Tr(M^{+}) - \lambda \cdot Tr(M^{-})$$

$$M \geq 0 \Longrightarrow \mathcal{M}^{-}_{\lambda,\Lambda}(M) = \lambda \cdot Tr(M) \;\&\; \mathcal{M}^{+}_{\lambda,\Lambda}(M) = \Lambda \cdot Tr(M)$$

$$A \in \mathcal{S}^{n\times n},\; spec(A) = \sigma(A) = \left\{\mu;\; \mu \text{ is an eigenvalue of } \; A\right\}$$

$$\mathcal{A}_{\lambda,\Lambda} := \left\{A \in \mathcal{S}^{n\times n};\; \mu \in \sigma(A) \Rightarrow \mu \in [\lambda,\Lambda]\right\}$$

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} Trace(AM), \quad \mathcal{M}^{+}_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} Trace(AM)$$

Envelope for Linear Equations

$$u \in C^2(B_1), \quad (UE) \;\; A(x) \in \mathcal{A}_{\lambda,\Lambda}, \; \forall x \in B_1.$$

$$Lu(x) := Tr(A(x)D^2u) = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j}(x) \qquad (\star)$$

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u(x)) \leq Lu(x) \leq \mathcal{M}^+_{\lambda,\Lambda}(D^2u(x))$$

$\forall L \;\; (UE)$ operator as in $(\star)$

$$Lu \leq f \text{ in } B_1 \Longrightarrow \mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f \text{ in } B_1$$

$$Lu \geq f \text{ in } B_1 \Longrightarrow \mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq f \text{ in } B_1$$

Equations in blue are Fully Nonlinear Elliptic PDEs

Pucci Operators for Nonnegative Matrices

$$0 < \lambda < \Lambda, \quad M \in \mathcal{S}^{n\times n}$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \lambda \cdot Tr(M^+) - \Lambda \cdot Tr(M^-)$$

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$$M \geq 0 \Longrightarrow \mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \cdot Tr(M) \;\&\; \mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \cdot Tr(M)$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^-_{\lambda,\Lambda}(N) \leq \mathcal{M}^-_{\lambda,\Lambda}(M+N) \quad \text{(superadditive)}$$

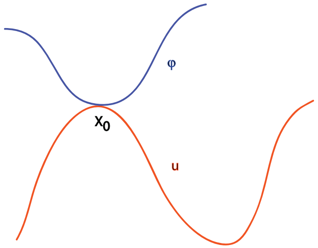
$$\mathcal{M}^-_{\lambda,\Lambda}(-M) = -\mathcal{M}^+_{\lambda,\Lambda}(M)$$

Viscosity Solution

Viscosity Solution (Motivation-I)

$u \in C^2(B_1)$ with $\Delta u \geq 0$ in B_1.

Assume that $\varphi \in C^2(B_\delta(x_0))$ touches u by above at x_0



Then, since $(u - \varphi)$ has a local maximum at x_0 then

$$D^2\varphi(x_0) \geq D^2 u(x_0) \Longrightarrow \Delta\varphi(x_0) \geq \Delta u(x_0) \geq 0$$

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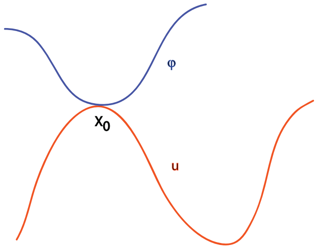

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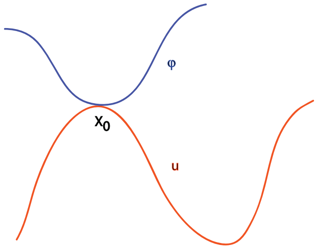

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Viscosity Solution (Motivation-II)

Suppose now $F : \mathcal{S}^{n\times n} \times B_1 \to \mathbb{R}$ so that

(Monotonicity) $\quad \forall x \in B_1,\ M \geq N$ in $\mathcal{S}^{n\times n} \Longrightarrow F(M,x) \geq F(N,x)$

$u \in C^2(B_1)$ with $F(D^2u, x) \geq 0$ in B_1.

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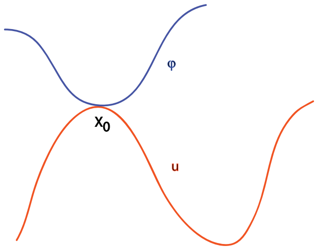

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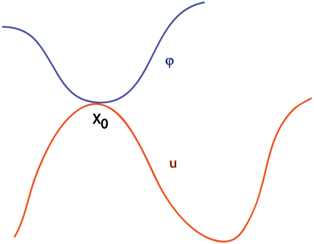

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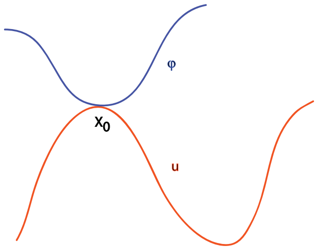

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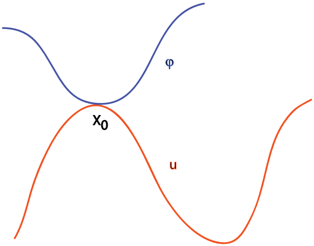

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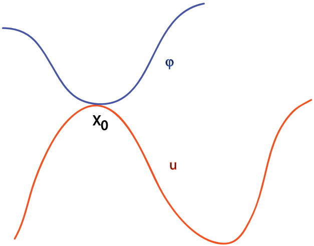

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