Nonlocal minimal surfaces

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Mostly Maximum Principle – June 2, 2022

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Outline

Nonlocal minimal surfaces

Energy functional dealing with *"pointwise interactions"* between a given set and its complement

Main idea: the "surface tension" is the byproduct of long-range interactions

Implications: nonlocal phase transitions and nonlocal capillarity theories

New effects due to the long-range interactions

Contributions from "far-away" can have a significant influence on the local structures of these new objects

STICKINESS Differently from classical minimal surfaces, the nonlocal minimal surfaces have the **strong tendency to "stick at the boundary"**

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The fractional perimeter functional

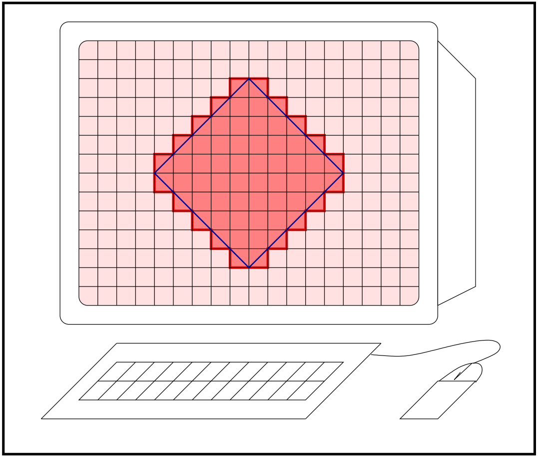
Given $s \in (0,1)$ and a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1,\gamma}$-boundary, the s-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in Ω is defined as

$$\begin{aligned}\mathrm{Per}_s(E;\Omega) &:= L(E\cap\Omega, (\mathcal{C}E)\cap\Omega)\\ &\quad + L(E\cap\Omega, (\mathcal{C}E)\cap(\mathcal{C}\Omega)) + L(E\cap(\mathcal{C}\Omega), (\mathcal{C}E)\cap\Omega),\end{aligned}$$

where $\mathcal{C}E = \mathbb{R}^n \setminus E$ denotes the complement of E, and $L(A,B)$ denotes the following nonlocal interaction term

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\, dx\, dy \qquad \forall A, B \subseteq \mathbb{R}^n,$$

This notion of s-perimeter and the corresponding minimization problem were introduced in [Caffarelli-Roquejoffre-Savin, 2010].

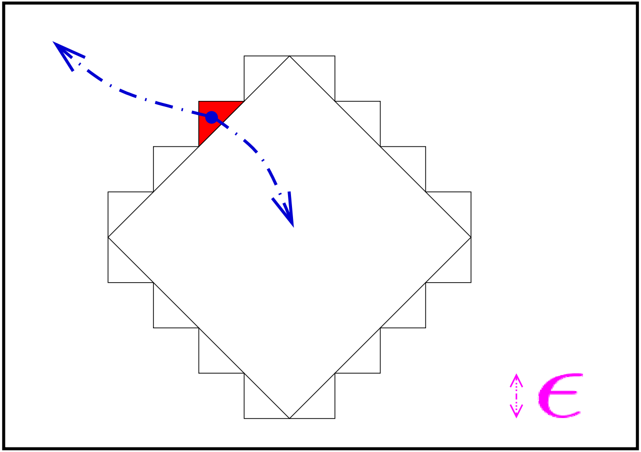


Side 1.

Perimeter 4.

Approximate Perimeter $4\sqrt{2}$.

Error $4(\sqrt{2}-1)$.



Error in each pixel $O(\epsilon^{2-s})$.
Number of pixels $O(\epsilon^{-1})$
Error $O(\epsilon^{1-s})$.

[Caffarelli-Roquejoffre-Savin, 2010]

1) Existence theorem:
there exists E s-minimizer for Per_s in Ω with $E \setminus \Omega = E_0 \setminus \Omega$.

2) Maximum principle:
E s-minimizer and $(\partial E) \setminus \Omega \subset \{|x_n| \leqslant a\} \Rightarrow \partial E \subset \{|x_n| \leqslant a\}$.

3) If ∂E is an hyperplane, then E is s-minimizer.

4) If E is s-minimizer in B_1, then ∂E is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set Σ, with Hausdorff dimension less or equal than $n-2$.

5) If E is s-minimizer and $0 \in \partial E$, then

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y|^{n+s}} dy = 0.$$

[Caffarelli-Roquejoffre-Savin, 2010]

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Regularity in dimension 2

[Savin-Valdinoci, 2013]:

Regularity of cones in dimension 2.

If E is s-minimizer in B_1, then ∂E is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set Σ, with Hausdorff dimension less or equal than $n-3$.

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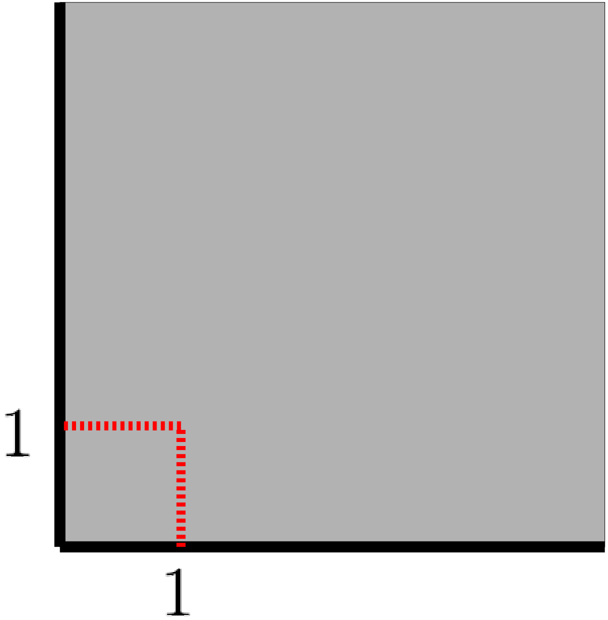
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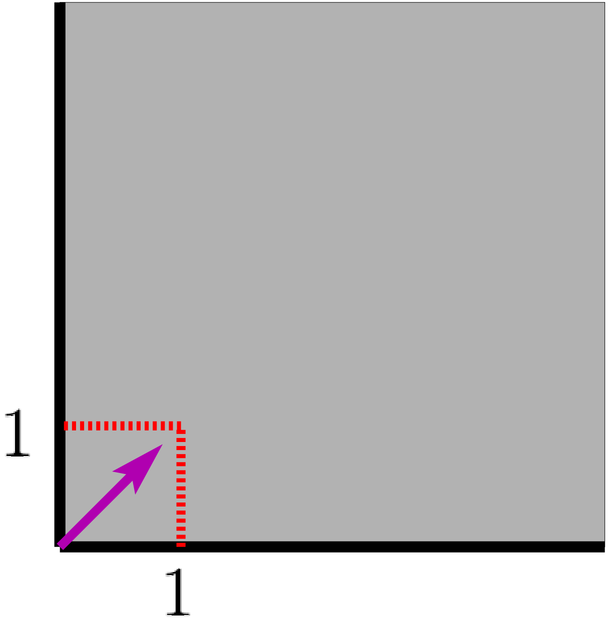
1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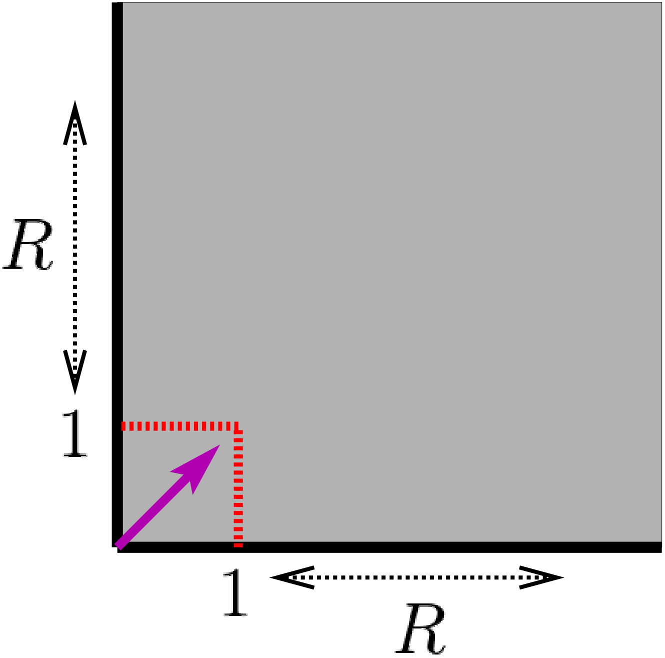
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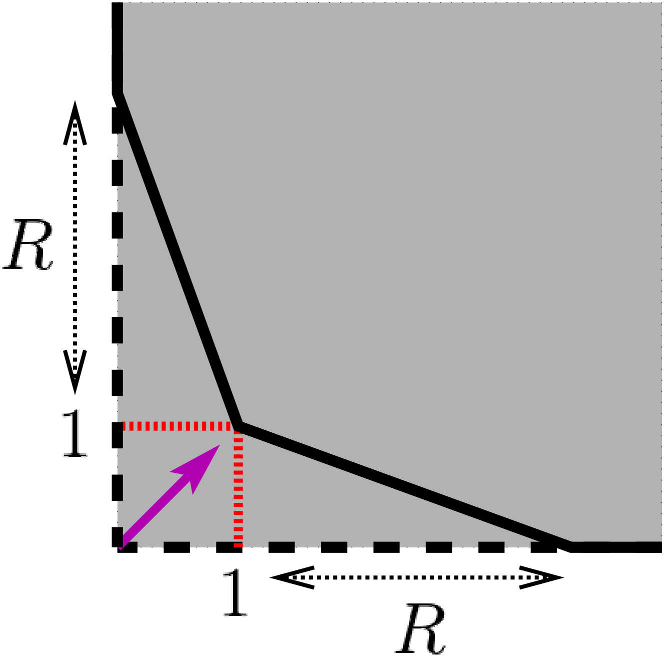
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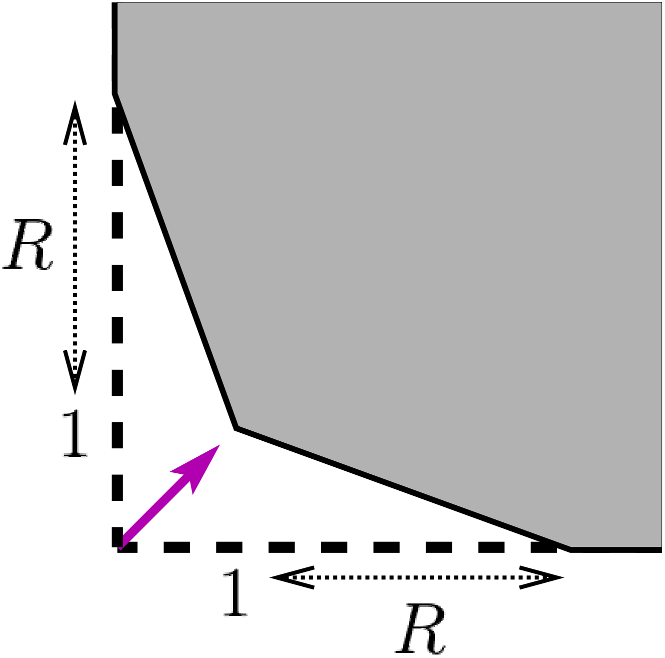

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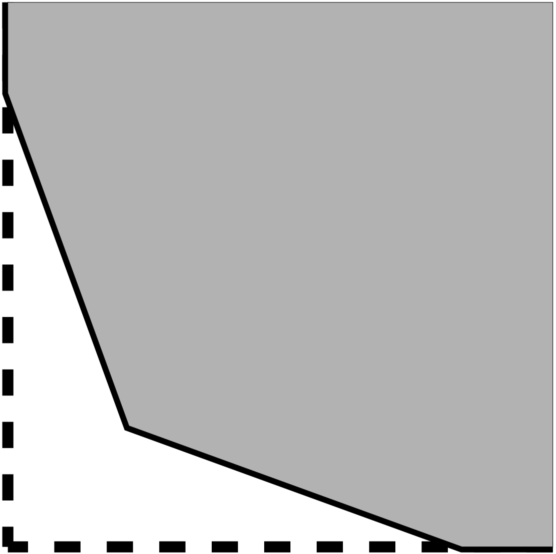

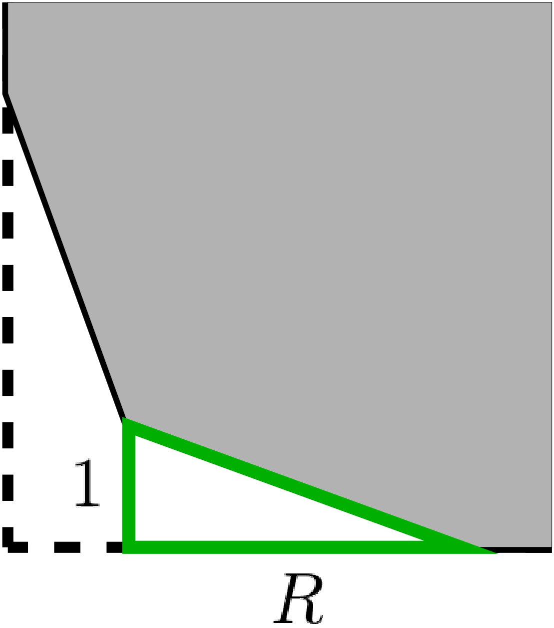
1
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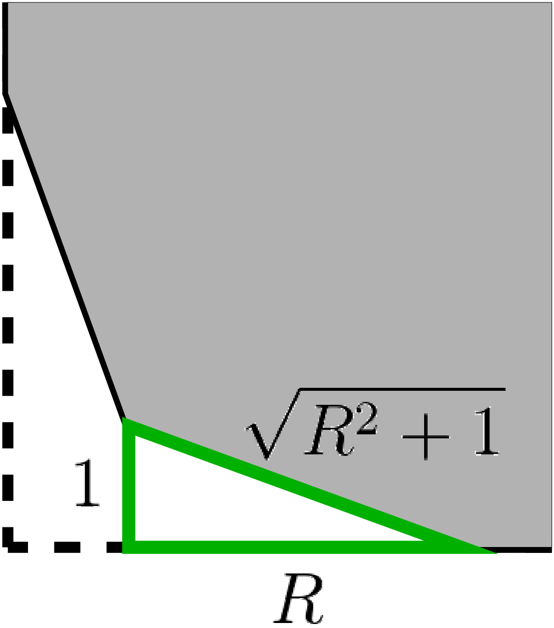
$\sqrt{R^2+1}$
1
R

Regularity for graphs in dimension 3

[Figalli-Valdinoci, 2013]:

Bernstein-type result:

- E is s-minimal in $\mathbb{R}^{n+1}$ and ∂E is a global graph,
- s-minimal surfaces are smooth in $\mathbb{R}^n$

$\Rightarrow \partial E$ is hyperplane.

Regularity of minimal graph in dimension 3.

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Regularity of minimal graph in dimension 3.

Limit as $s \to 1$

[Bourgain-Brezis-Mironescu, 2001], [Dávila, 2002], [Ponce, 2004], [Caffarelli-Valdinoci, 2011], [Ambrosio-De Philippis-Martinazzi, 2011], [Lombardini, 2018]:

$$(1-s)\mathrm{Per}_s \to \mathrm{Per}, \qquad \text{as } s \nearrow 1$$

(up to normalizing multiplicative constants).

$$\Downarrow$$

[Caffarelli-Valdinoci, 2013]:
s close to 1: nonlocal minimal surfaces are as regular as classical minimal surfaces.

(If E is s-minimizer in B_1, then ∂E is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set Σ, with Hausdorff dimension less or equal than $n-8$.)

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Limit as $s \to 0$

[Maz'ya-Shaposhnikova, 2002] and
[Dipierro-Figalli-Palatucci-Valdinoci, 2013]:
If there exists the limit

$$\alpha(E) := \lim_{s \searrow 0} s \int_{E \cap (\mathcal{C}B_1)} \frac{1}{|y|^{n+s}}\, dy,$$

then

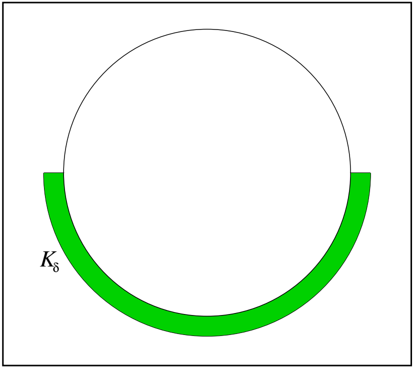
$$\lim_{s \searrow 0} s \operatorname{Per}_s(E, \Omega) = \big(\omega_{n-1} - \alpha(E)\big) \frac{|E \cap \Omega|}{\omega_{n-1}} + \alpha(E) \frac{|\Omega \setminus E|}{\omega_{n-1}}.$$

Stickiness to half-balls

For any $\delta > 0$,

$$K_\delta := \big(B_{1+\delta} \setminus B_1\big) \cap \{x_n < 0\}.$$

We define E_δ to be the set minimizing the s-perimeter among all the sets E such that $E \setminus B_1 = K_\delta$.



Stickiness to half-balls

There exists $\delta_o > 0$ such that for any $\delta \in (0, \delta_o]$ we have that

$$E_\delta = K_\delta.$$

Stickiness to the sides of a box

Given a large $M > 1$ we consider the s-minimal set E_M in $(-1,1) \times \mathbb{R}$ with datum outside $(-1,1) \times \mathbb{R}$ given by the jump $J_M := J_M^- \cup J_M^+$, where

$$J_M^- := (-\infty, -1] \times (-\infty, -M)$$
$$\text{and} \qquad J_M^+ := [1, +\infty) \times (-\infty, M).$$

Stickiness to the sides of a box

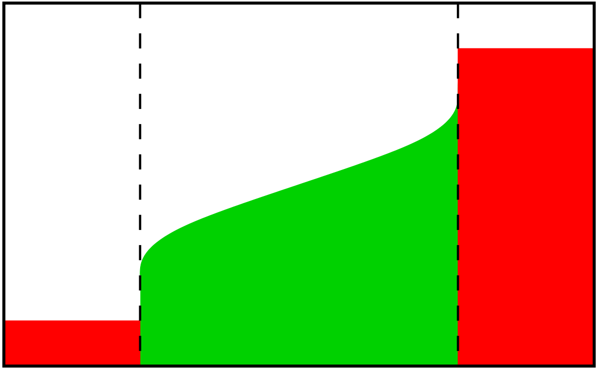
There exist $M_o > 0$ and $C_o \geq C'_o > 0$, depending on s, such that if $M \geq M_o$ then

$$[-1,1) \times [C_o M^{\frac{1+s}{2+s}},\, M] \subseteq E_M^c$$

$$\text{and} \qquad (-1,1] \times [-M,\, -C_o M^{\frac{1+s}{2+s}}] \subseteq E_M.$$

Also, the exponent $\beta := \frac{1+s}{2+s}$ above is optimal.

Stickiness to the sides of a box



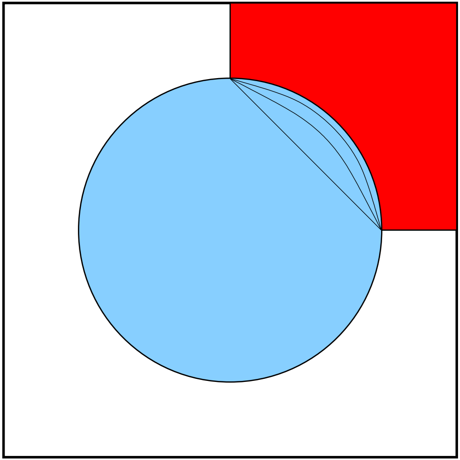
Stickiness as $s \to 0^+$

We consider a sector in $\mathbb{R}^2$ outside B_1, i.e.

$$\Sigma := \{(x, y) \in \mathbb{R}^2 \setminus B_1 \text{ s.t. } x > 0 \text{ and } y > 0\}.$$

Let E_s be the s-minimizer of the s-perimeter among all the sets E such that $E \setminus B_1 = \Sigma$.
Then, there exists $s_o > 0$ such that for any $s \in (0, s_o]$ we have that $E_s = \Sigma$.

Stickiness as $s \to 0^+$



Instability of the flat fractional minimal surfaces

Fix $\epsilon_0 > 0$ arbitrarily small. Then, there exists $\delta_0 > 0$, possibly depending on ϵ_0, such that for any $\delta \in (0, \delta_0]$ the following statement holds true.

Assume that $F \supset H \cup F_- \cup F_+$, where

$$H := \mathbb{R} \times (-\infty, 0),$$

$$F_- := (-3, -2) \times [0, \delta)$$

and

$$F_+ := (2, 3) \times [0, \delta).$$

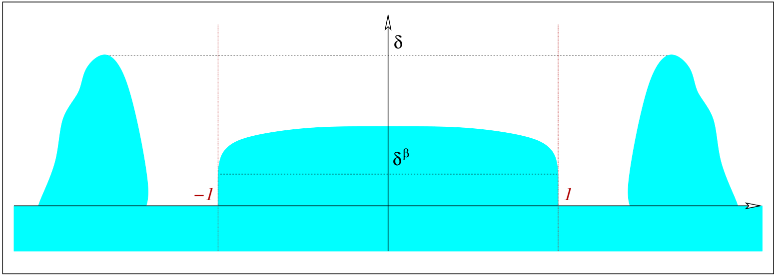
Let E be the s-minimal set in $(-1, 1) \times \mathbb{R}$ among all the sets that coincide with F outside $(-1, 1) \times \mathbb{R}$.

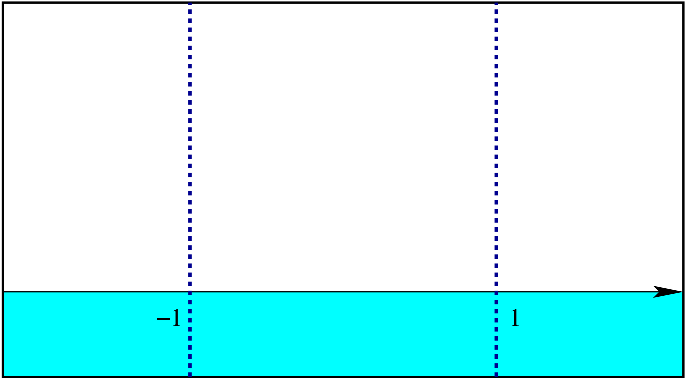
Then

$$E \supseteq (-1, 1) \times (-\infty, \delta^{\frac{2+\epsilon_0}{1-s}}].$$

Instability of the flat fractional minimal surfaces

$$\beta := \frac{2+\epsilon_0}{1-s}$$





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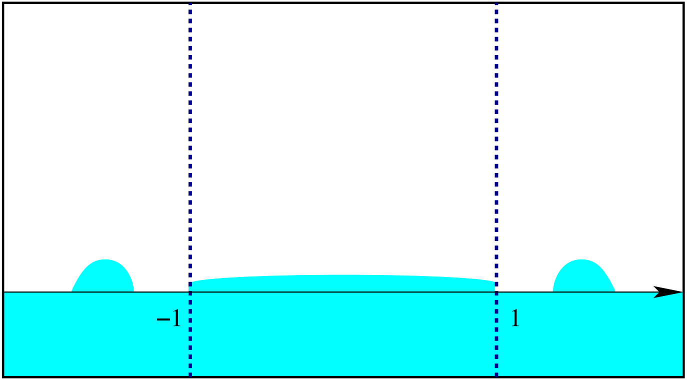

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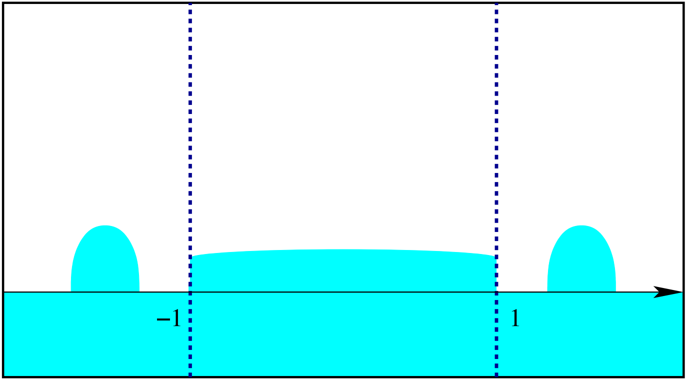

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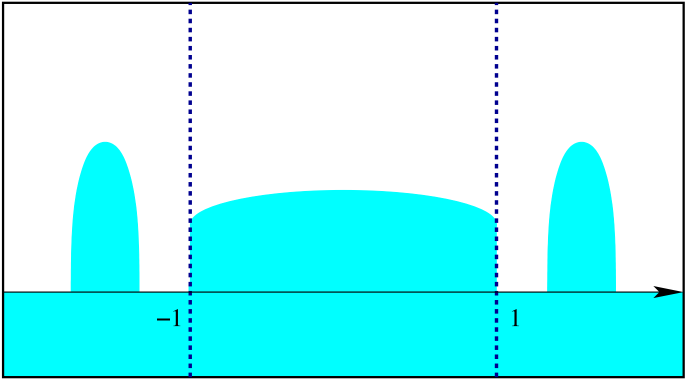

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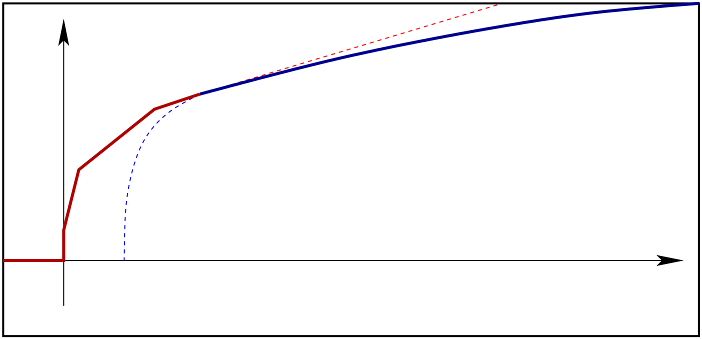
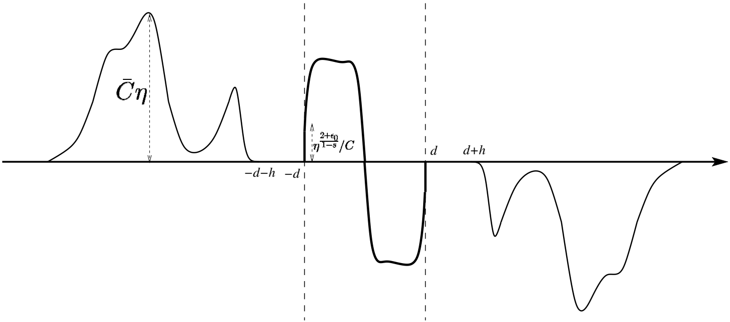
A useful barrier

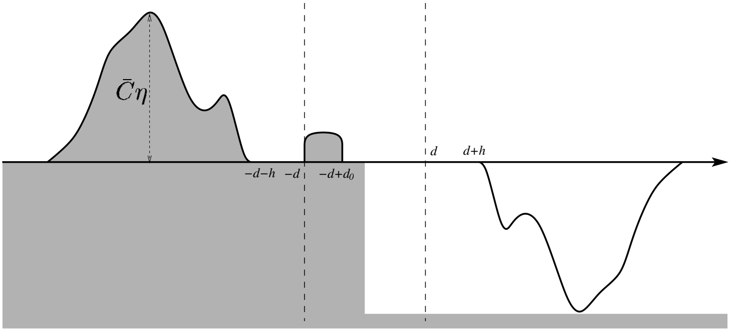


Stickiness for antisymmetric data [Baronowitz-Dipierro-Valdinoci, 2022]



Stickiness for antisymmetric data [Baronowitz-Dipierro-Valdinoci, 2022]

Use a barrier of this type:



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Theorem

Let $d,\ h > 0$. Let $u : \mathbb{R} \to \mathbb{R}$ be an s-minimal graph in $(-d, d)$, with $u \in C(-\infty, -d) \cap C^{1,\frac{1+s}{2}}(-d-h, -d)$. Assume that

$$\begin{aligned} & u(x) = -u(-x) \quad \textit{for all } x \in (d, +\infty) \\ \textit{and} \quad & u(x) \le 0 \quad \textit{for all } x \in (d, +\infty). \end{aligned}$$

Then,

$$\begin{aligned} & u(x) = -u(-x) \quad \textit{for all } x \in (-\infty, +\infty), \\ & u(x) \ge 0 \quad \textit{for all } x \in (-\infty, 0] \\ \textit{and} \quad & u(x) \le 0 \quad \textit{for all } x \in [0, +\infty). \end{aligned}$$

Three further questions [Dipierro-Savin-Valdinoci, 2020]

1. How regular are the nonlocal minimal surfaces *coming from inside the domain*?

2. Is the Euler-Lagrange equation satisfied *up to the boundary*?

3. How *typical* is the stickiness phenomenon?

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Regularity coming from inside

"Continuity implies differentiability"

Consider a nonlocal minimal graph in $(0,1)$, with a smooth external graph u_0.

There is a dichotomy:

- either

$$\lim_{x\nearrow 0} u_0(x) \neq \lim_{x\searrow 0} u(x)$$

  and

$$\lim_{x\searrow 0} |u'(x)| = +\infty,$$

- or

$$\lim_{x\nearrow 0} u_0(x) = \lim_{x\searrow 0} u(x)$$

  and u is $C^{1,\frac{1+s}{2}}$ at 0.

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and u is $C^{1,\frac{1+s}{2}}$ at 0.

Some remarks

This dichotomy is a purely nonlinear effect, since the boundary behavior of linear equation is of Hölder type [Serra-Ros Oton].

Some remarks

Stickiness + dichotomy = butterfly effect

An arbitrarily small perturbation of the flat data produce a boundary discontinuity, which entails an infinite derivative at the boundary.

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As a curve, the nonlocal minimal graph turns out to be always $C^{1,\frac{1+s}{2}}$:

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Boundary Euler-Lagrange equations

If u is a nonlocal minimal graph in $(0,1)$ with smooth datum outside, then

$$\int_{-\infty}^{+\infty} F\left(\frac{u(x+y)-u(x)}{|y|}\right)\frac{dy}{|y|^{1+s}} = 0$$

for all $x \in [0,1]$.

Stickiness is generic

Let $\varphi \in C_0^\infty([-2,-1],\ [0,1])$, with $\varphi \not\equiv 0$.

Let $u^{(t)}$ be the nonlocal minimal graph in $(0,1)$ with external datum

$$u_0^{(t)} := u_0 + t\varphi.$$

Suppose that

$$\lim_{x \nearrow 0} u_0(x) = \lim_{x \searrow 0} u(x).$$

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Proof of dichotomy

Think about the usual suspects (discontinuous, Lipschitz, Hölder, smooth).

Blow-up.

The "worst" cases to understand are the Hölder and the smooth (the Lipschitz produces non-minimal corners).

The smooth case produces flat objects: use a boundary improvement of flatness (combined with a boundary monotonicity formula) to deduce smoothness of the initial minimizer (for this, use new barrier to go beyond the linear theory!).

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We consider nonlocal minimal surfaces in a cylinder with prescribed datum given by the complement of a slab.

$$\Omega := \{(x', x_n) \text{ with } |x'| < 1\}.$$

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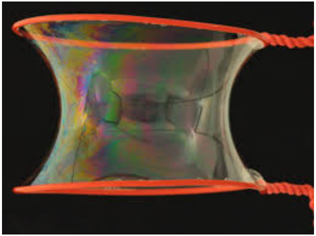

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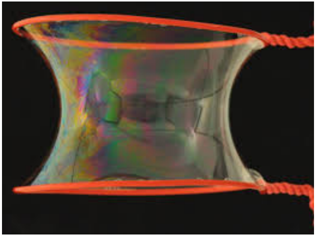

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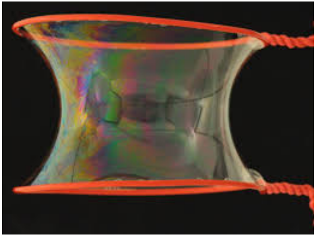

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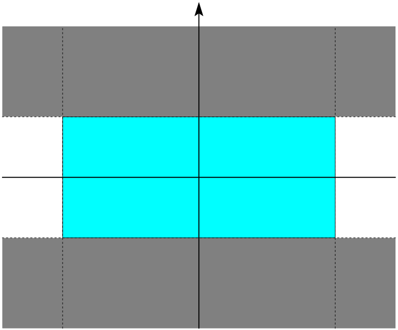

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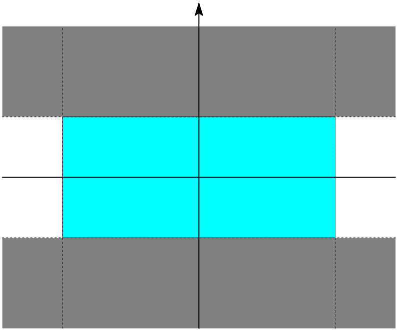

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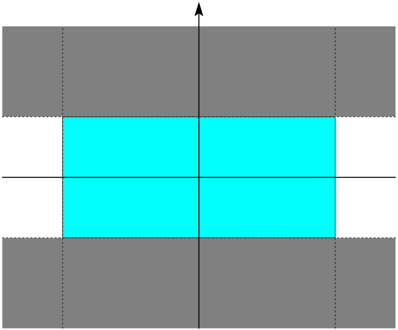

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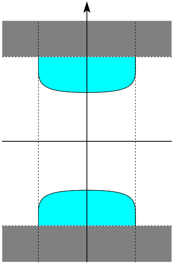
so it is not the complement of a slab. Also (at least in dimension 2) it sticks at the boundary.



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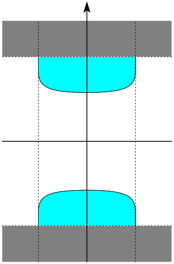

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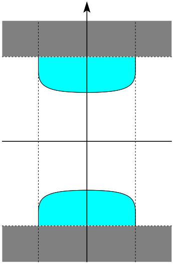

Yin-Yang Theorems

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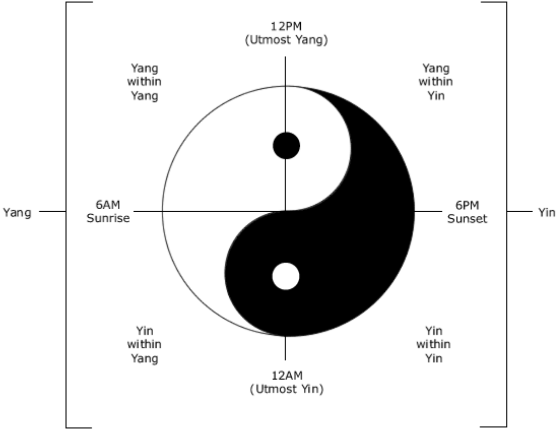
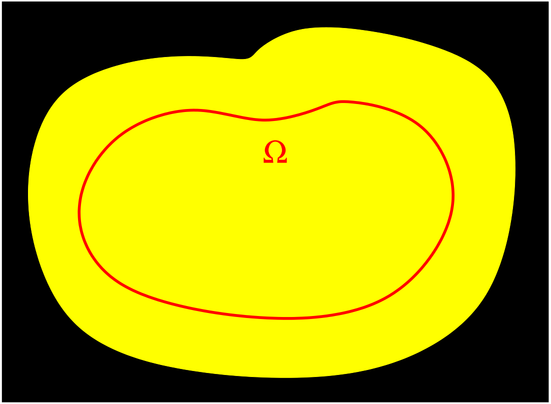


Yin-Yang Theorems [Bucur-Dipierro-Lombardini-Valdinoci, 2020]

There exists $\vartheta > 1$ such that if E is s-minimal in $\Omega \subset \mathbb{R}^n$ and $E \cap (\Omega_{\vartheta \operatorname{diam}(\Omega)} \setminus \Omega) = \varnothing$, then

$$E \cap \Omega = \varnothing.$$



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Let u be s-minimal in $(-1,1)\times(0,1)\times\mathbb{R}$ with $u=0$ in $(-2,2)\times\left(-\frac{1}{100},0\right)$.

Consider the trace of u

$$f(x) := \lim_{y\searrow 0} u(x,y).$$

Assume that $f(0)=0$. Then, near the origin,

$$|u(x,y)| \le C\left(x^2+y^2\right)^{\frac{3+s}{4}}.$$

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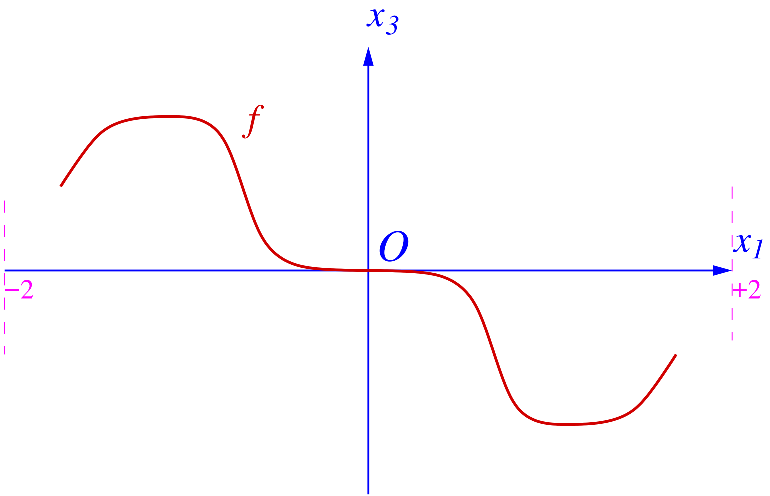
Vanishing of the gradient of the trace at the zero crossing points



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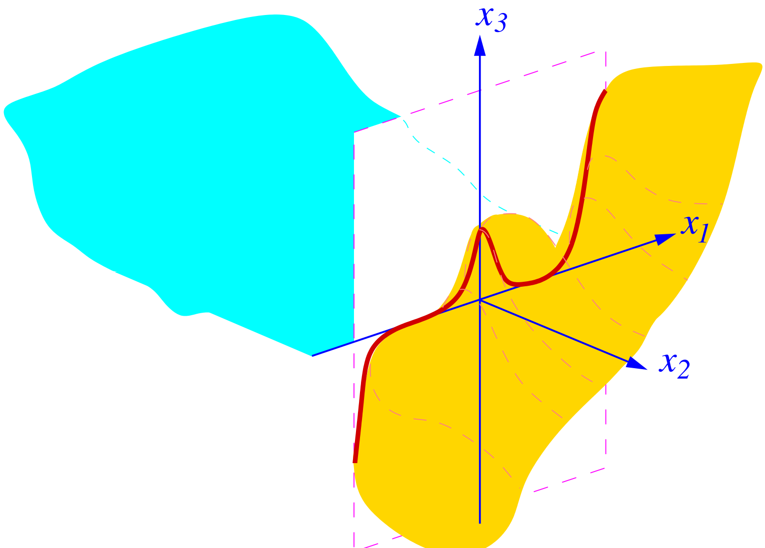

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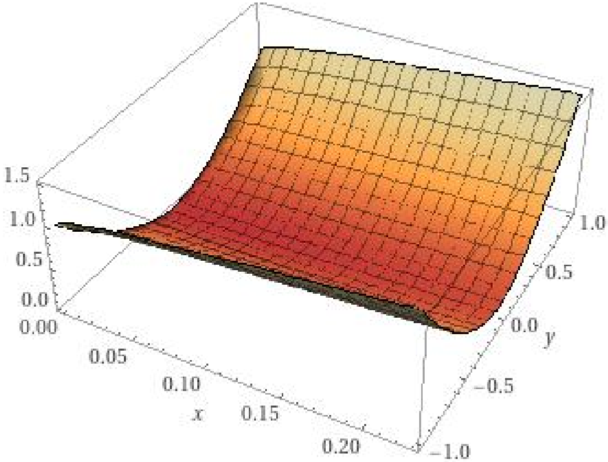

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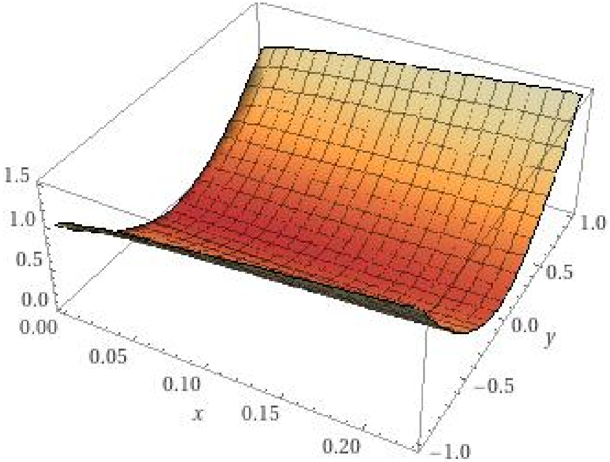

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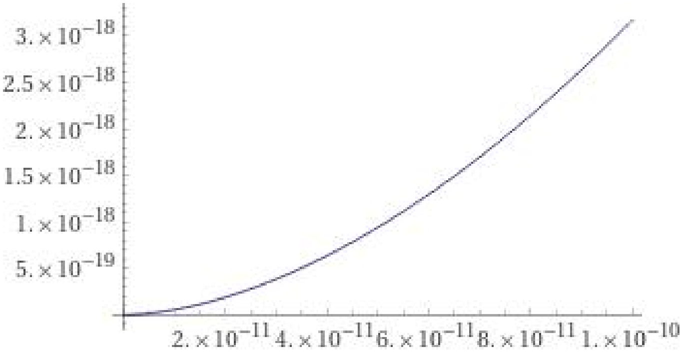

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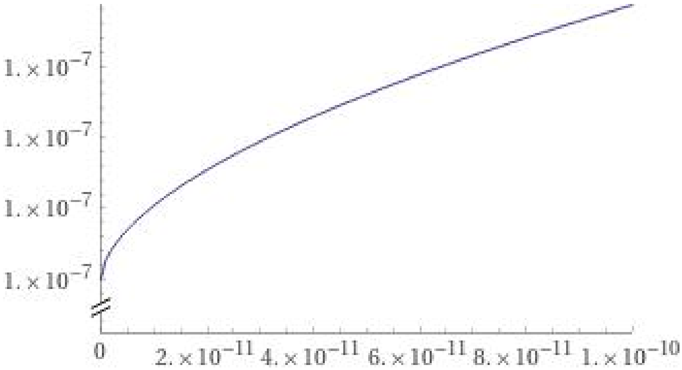

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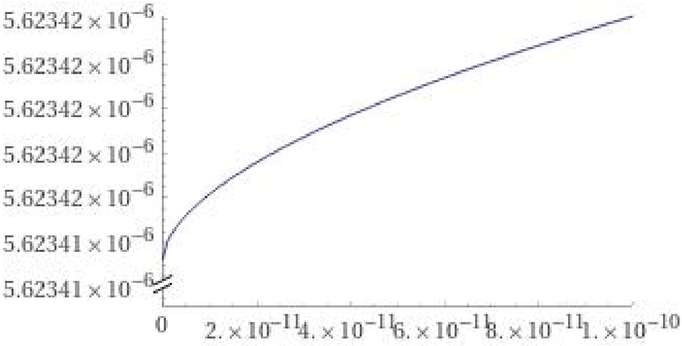

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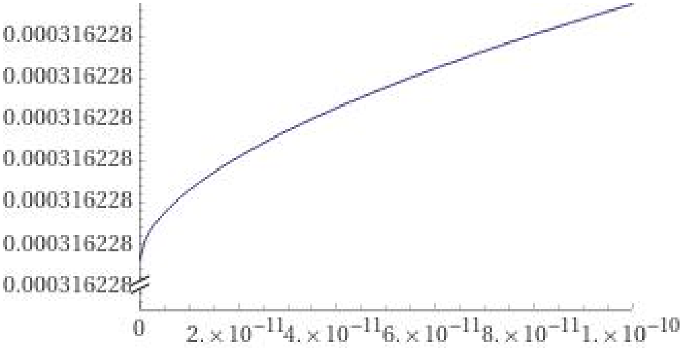

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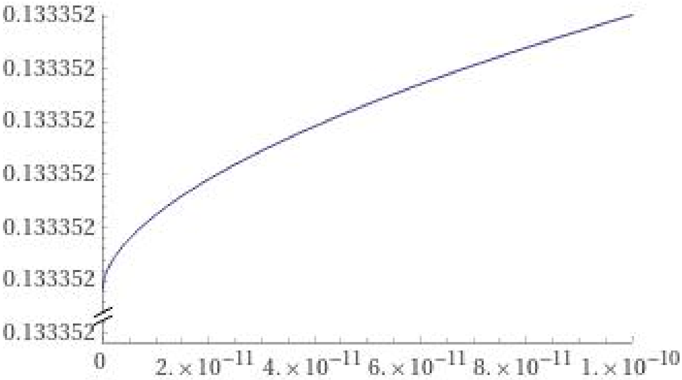

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Stickiness in dimension 3 [Dipierro-Savin-Valdinoci, 2020]

Pivotal step: if a homogeneous nonlocal minimal graph in $\{x > 0\}$ vanishes in $\{x < 0\}$ and is continuous at the origin, then it necessarily vanishes at all points:

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be an s-minimal graph in $\{x > 0\}$, with $u = 0$ in $\{x < 0\}$.

Assume also that u is positively homogeneous of degree 1, i.e. $u(tX) = tu(X)$ for all $X \in \mathbb{R}^2$ and $t > 0$. Suppose that

$$\lim_{x \searrow 0} u(x, y) = 0.$$

Then $u \equiv 0$.

Open problems [Dipierro-Savin-Valdinoci, 2020]

What happens in dimension $n \geq 4$?

(Dimension 3 was "easier" because the trace is a function from $\mathbb{R}$ to $\mathbb{R}$, so knowing the derivative at a point, together with the 1-homogeneity, determines already half of the trace; in dimension 4 this only determines the trace along a half-line).

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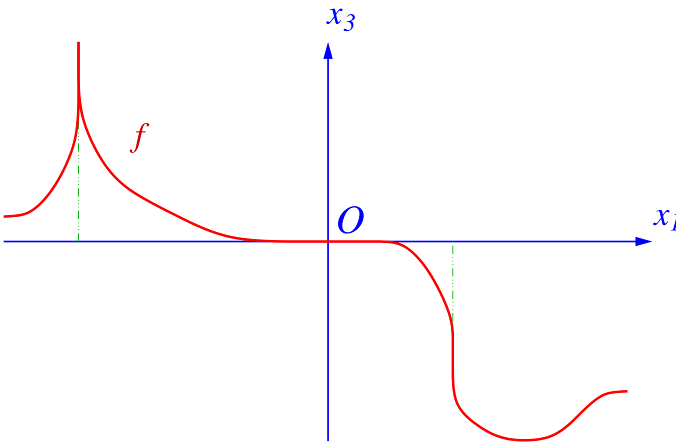

Is it possible to construct examples of nonlocal minimal graphs which are locally flat from outside and whose trace develops vertical tangencies?

Open problems [Dipierro-Savin-Valdinoci, 2020]

What is the behavior of a nonlocal minimal graph and of its trace at the corners of the domain and in their vicinity?

Can one understand (dis)continuity and tangency properties, possibly also in relation with the convexity or concavity of the corner?

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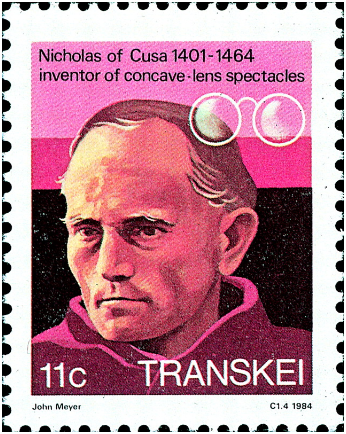
If full knowledge about the very base of our existence could be described as a circle, the best we can do is to arrive at a polygon.



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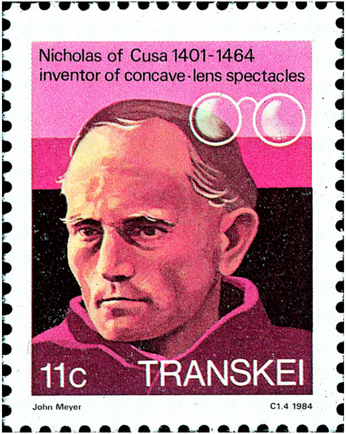

Nicholas of Cusa

How “nonlinear” is the problem?

The linearization of the trace of a nonlocal minimal graph is given by the fractional normal derivative of a fractional Laplace problem.

Indeed, if u is a nonlocal minimal graph, say in $x \in (0,1)$, and it is ε-flat near the origin, then $\frac{u}{\varepsilon}$ (the “vertical rescaling”) tends to a function $\overline{u}$ which is a solution of $(-\Delta)^{\frac{1+s}{2}}\overline{u}(x) = 0$ for $x \in (0,1)$.

By the boundary regularity of linear equation (Serra, Ros-Oton, Grubb, etc.) the first order of $\overline{u}$ is of Hölder type: near the origin $\overline{u} \simeq ax^{\frac{1+s}{2}}$, for some $a \in \mathbb{R}$.

So, one may expect that, near the origin, $u(x) \simeq a\varepsilon x^{\frac{1+s}{2}}$.

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Flexibility of linear equations [Dipierro-Savin-Valdinoci, 2020]

But this is not the case! The fractional normal derivative of a fractional Laplace problem is not only different than zero in general, but it can be arbitrarily prescribed:

Let $n \geq 2$ and $f \in C(\mathbb{R}^{n-1})$. Then, for every $\delta > 0$ there exist $f_\delta, u_\delta \in C(\mathbb{R}^{n-1})$ such that

$$\begin{cases} \sup_{|x'|\leq 1} |f_\delta(x') - f(x')| \leq \delta, \\ (-\Delta)^\sigma u_\delta = 0 \text{ in } B_1 \cap \{x_n > 0\}, \\ u_\delta = 0 \text{ in } \{x_n < 0\}, \\ \lim_{x_n \searrow 0} \frac{u_\delta(x)}{x_n^\sigma} = f_\delta(x') \text{ for all } |x'| < 1. \end{cases}$$

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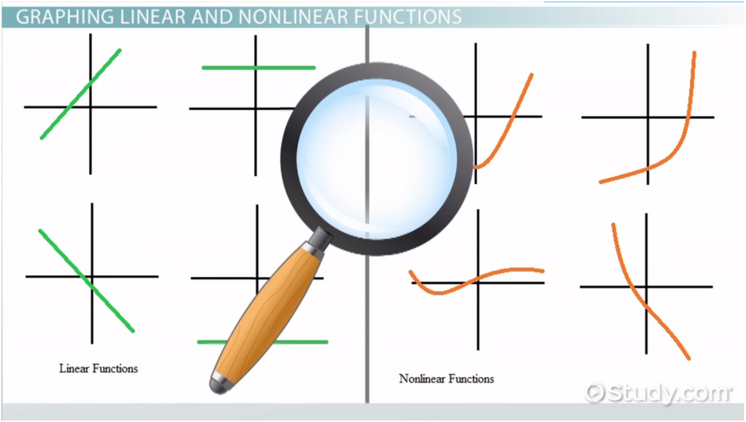

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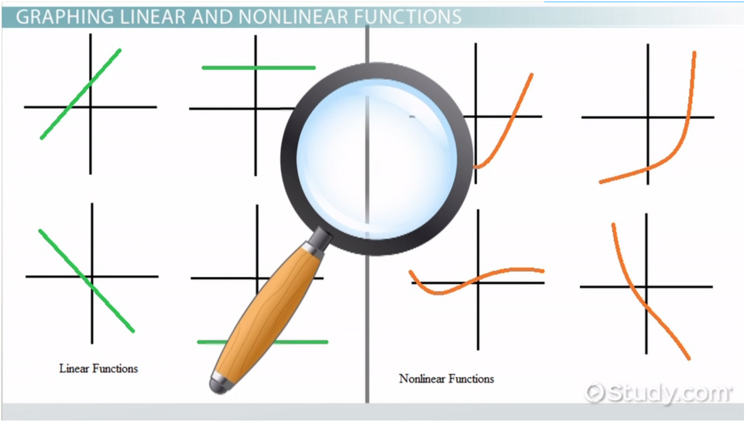

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Thank you very much for your attention!