Some New Insights on the Maximum Principle for Higher Order Operators

Daniele Cassani

Università degli Studi dell'Insubria
and
Riemann International School of Mathematics
`https://www.rism.it`

June 3rd, 2022

In loving memory of Louis

Boggio, 1905

Let $m \geq 1$ and $u \in L^1(B_r)$ be such that

$$\int_{B_r} u(x)\, \Delta^{2m}\varphi(x)\ dx \geq 0, \qquad \forall \varphi \in C^{4m}(B_r) \cap H_0^2(B_r),\ \ \varphi \geq 0.$$

Then $u \geq 0$. Moreover $u > 0$ in B_r or $u \equiv 0$ a.e. in B_r.

Boggio, 1905

Let $m \geq 1$ and $u \in L^1(B_r)$ be such that

$$\int_{B_r} u(x)\,\Delta^{2m}\varphi(x)\ dx \geq 0, \quad \forall \varphi \in C^{4m}(B_r) \cap H_0^2(B_r),\ \ \varphi \geq 0.$$

Then $u \geq 0$. Moreover $u > 0$ in B_r or $u \equiv 0$ a.e. in B_r.

So far the strong maximum principle holds for the following operators:

- $N = 2$, $\Delta^{2m} + \varepsilon L$, $4m$-uniformly elliptic small perturbation on slight deformations of the ball;
- $N \geq 2$, Δ^{2m}, $m \geq 1$ on slight deformations of the ball;
- $N \geq 2$, $\Delta^{2m} + \sum_{|\alpha| \leq 4m-1} a_\alpha(x) D^\alpha u$, where a_α is small enough, on the ball.

F. Gazzola, H. C. Grunau, G. Sweers, *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Springer-Verlag, Berlin, 2010.

Bad news...

Grunau-Sweers, 2014

MP is false without further assumptions even for Δ^{2m}. For any $N \geq 2$, there is a bounded smooth domain $\Omega \subset \mathbb{R}^N$ such that the solution of $\Delta^2 u = 1$ on Ω and $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, changes sign.

Bad news...

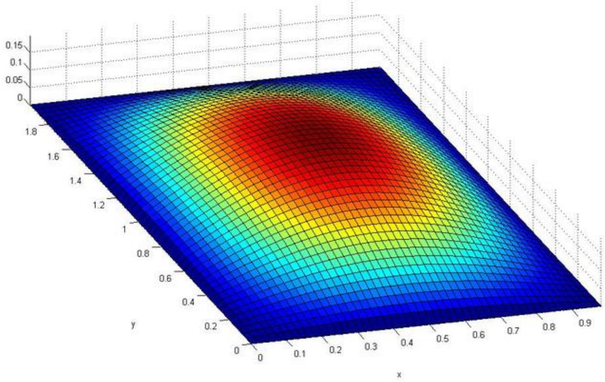
Grunau-Sweers, 2014

MP is false without further assumptions even for Δ^{2m}. For any $N \geq 2$, there is a bounded smooth domain $\Omega \subset \mathbb{R}^N$ such that the solution of $\Delta^2 u = 1$ on Ω and $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, changes sign.

Abatangelo-Jarohs-Saldana, 2018

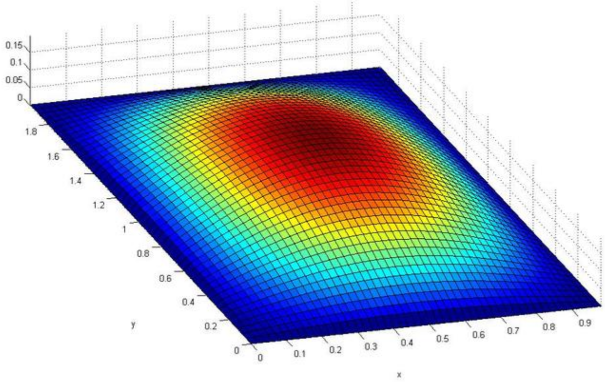
MP fails also for the higher order fractional Laplacian.

Heuristic: bending vs tension



$$E(u) = \frac{1}{2}\int |\Delta u|^2 + \frac{\gamma}{2}\int |\nabla u|^2$$

Heuristic: bending vs tension



$E(u) = \frac{1}{2} \int |\Delta u|^2 + \frac{\gamma}{2} \int |\nabla u|^2$

Do we miss something?

Two key ingredients from second order elliptic equations:

- Caccioppoli's inequality;
- Harnack's inequality.

Caccioppoli's inequality

Let $u \in W^{1,2}(\Omega)$ be a solution of

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) = 0, \tag{0.1}$$

where the matrix of $L^\infty(\Omega)$ coefficients is uniformly elliptic on a open set $\Omega \subset \mathbb{R}^N$. Then, there exists $c > 0$ such that

$$\int_{A^+(x_0,k,\rho)} |\nabla u(x)|^2 \, dx \le \frac{c}{(r-\rho)^2} \int_{A^+(x_0,k,r)} |u(x) - k|^2 \, dx, \quad 0 < \rho < r \tag{0.2}$$

where $A^+(x_0, k, r) = \{x : \ x \in B(x_0, r), \ \ u(x) > k\}$.

Harnack's inequality and consequences

Definition

We say that $u : \Omega \longrightarrow \mathbb{R}$ *satisfies Harnack's inequality if there exists* $c > 0$ *such that for all* $B(x_0, R) \subset \Omega$ *we have*

$$\sup_{B(x_0,r)} u \leq c \inf_{B(x_0,R)} u, \quad r \leq R. \tag{0.3}$$

Strong maximum principle

Let $u \geq 0$ satisfy (0.3). Then $u > 0$ on Ω or $u \equiv 0$ on Ω.

Just apply (0.3) to $u + \varepsilon$, for all $\varepsilon > 0$.

Regularity

If u satisfies (0.3) then it is Hölder continuous on Ω.

Harnack's inequality and consequences

Definition

We say that $u : \Omega \longrightarrow \mathbb{R}$ *satisfies Harnack's inequality if there exists* $c > 0$ *such that for all* $B(x_0, R) \subset \Omega$ *we have*

$$\sup_{B(x_0,r)} u \leq c \inf_{B(x_0,R)} u, \qquad r \leq R. \tag{0.3}$$

Strong maximum principle

Let $u \geq 0$ satisfy (0.3). Then $u > 0$ on Ω or $u \equiv 0$ on Ω.

Just apply (0.3) to $u + \varepsilon$, for all $\varepsilon > 0$.

Regularity

If u satisfies (0.3) then it is Hölder continuous on Ω.

Remark. We do not require u to solve a PDE.

Harnack's inequality and consequences

Definition

We say that $u : \Omega \longrightarrow \mathbb{R}$ satisfies Harnack's inequality if there exists $c > 0$ such that for all $B(x_0, R) \subset \Omega$ we have

$$\sup_{B(x_0,r)} u \leq c \inf_{B(x_0,R)} u, \quad r \leq R. \tag{0.3}$$

Strong maximum principle

Let $u \geq 0$ satisfy (0.3). Then $u > 0$ on Ω or $u \equiv 0$ on Ω.

Just apply (0.3) to $u + \varepsilon$, for all $\varepsilon > 0$.

Regularity

If u satisfies (0.3) then it is Hölder continuous on Ω.

Remark. We do not require u to solve a PDE.

Then: what are the functions which satisfy (0.3)?

Answer:

- *Subharmonic positive functions*: $u \in C^2(\Omega)$ s.t. $\Delta u \geq 0$ or (by Weyl's lemma) $u \in C^0(\Omega)$ s.t.

$$\forall \varphi \in C_C^\infty(\Omega), \;\; \varphi \geq 0 : \int_\Omega u(x)\Delta\varphi(x) \;\, dx \geq 0.$$

- $u \in W^{1,2}(\Omega)$ which are *positive solutions* to

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) \;=\; 0, \tag{0.4}$$

where the matrix of $L^\infty(\Omega)$ coefficients is uniformly elliptic on Ω.

- $u \in W^{1,2}(\Omega)$, $u \geq 0$ belongs to $DG(\Omega)$, *De Giorgi's class* on Ω: $\exists\, c > 0$ s.t.$\forall x_0 \in \Omega$, $B(x_0, r) \subset \Omega$ and $\forall k \in \mathbb{R}$

$$\int_{A^+(x_0,k,\rho)} |\nabla u(x)|^2 \, dx \;\leq\; \frac{c}{(r-\rho)^2} \int_{A^+(x_0,k,r)} |u(x) - k|^2 \; dx, \quad 0 < \rho < r \tag{0.5}$$

E. Di Benedetto, N. S. Trudinger, *Harnack inequalities for quasi-minima of variational integrals*, Ann. Inst. Henri Poincaré, Analyse Non Lineairé, **1** (4), (1984), 295-308.

Answer:

- *Subharmonic positive functions*: $u \in C^2(\Omega)$ s.t. $\Delta u \geq 0$ or (by Weyl's lemma) $u \in C^0(\Omega)$ s.t.

$$\forall \varphi \in C_C^\infty(\Omega), \ \ \varphi \geq 0 : \int_\Omega u(x)\Delta\varphi(x) \ \ dx \geq \ 0.$$

- $u \in W^{1,2}(\Omega)$ which are *positive solutions* to

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) \ = \ 0, \tag{0.4}$$

where the matrix of $L^\infty(\Omega)$ coefficients is uniformly elliptic on Ω.

- $u \in W^{1,2}(\Omega)$, $u \geq 0$ belongs to $DG(\Omega)$, *De Giorgi's class* on Ω: $\exists \ c > 0$ s.t.$\forall x_0 \in \Omega$, $B(x_0, r) \subset \Omega$ and $\forall k \in \mathbb{R}$

$$\int_{A^+(x_0,k,\rho)} |\nabla u(x)|^2 \, dx \ \leq \ \frac{c}{(r-\rho)^2} \int_{A^+(x_0,k,r)} |u(x) - k|^2 \ dx, \quad 0 < \rho < r \tag{0.5}$$

E. Di Benedetto, N. S. Trudinger, *Harnack inequalities for quasi-minima of variational integrals*, Ann. Inst. Henri Poincaré, Analyse Non Lineairé, **1** (4), (1984), 295-308.

Remark. Solutions to (0.4) do have membership in $DG(\Omega)$.

A main obstruction in the higher order context:

If $u \in W^{1,2}(\Omega)$ solves

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) \;=\; 0,$$

then $(u-k)^+ \;=\; max\{(u-k), 0\} \in W^{1,2}(\Omega)$ is also a solution.
In contrast, if $u \in W^{m,2}(\Omega)$, $m > 1$, solves

$$\sum_{|\alpha|=|\beta|=m} D^{\alpha}(a_{\alpha\beta}(x)D^{\beta}u(x)) = 0$$

in general $(u-k)^+ \notin W^{m,2}(\Omega)$.

A main obstruction in the higher order context:

If $u \in W^{1,2}(\Omega)$ solves

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) \;=\; 0,$$

then $(u-k)^+ \;=\; max\{(u-k), 0\} \in W^{1,2}(\Omega)$ is also a solution.
In contrast, if $u \in W^{m,2}(\Omega)$, $m > 1$, solves

$$\sum_{|\alpha|=|\beta|=m} D^{\alpha}(a_{\alpha\beta}(x)D^{\beta}u(x)) = 0$$

in general $(u-k)^+ \notin W^{m,2}(\Omega)$.

However, functions enjoying Harnack's inequality satisfy the MP with no need for being solutions to PDE nor belonging to $DG(\Omega)$.

A main obstruction in the higher order context:

If $u \in W^{1,2}(\Omega)$ solves

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) \;=\; 0,$$

then $(u-k)^+ \;=\; max\{(u-k), 0\} \in W^{1,2}(\Omega)$ is also a solution.
In contrast, if $u \in W^{m,2}(\Omega)$, $m > 1$, solves

$$\sum_{|\alpha|=|\beta|=m} D^{\alpha}(a_{\alpha\beta}(x)D^{\beta}u(x)) = 0$$

in general $(u-k)^+ \notin W^{m,2}(\Omega)$.

However, functions enjoying Harnack's inequality satisfy the MP with no need for being solutions to PDE nor belonging to $DG(\Omega)$.

Philosophy

Looking for a Harnack type inequality for functions which do not necessarily belong to $DG(\Omega)$ though with augmented regularity, namely $W^{1,t}(\Omega)$, $t > N$.

A Harnack type inequality with remainder term

Theorem (C.–Tarsia, 2021)

Let $u \in W^{1,t}(\Omega)$, where $t > N \geq 2$. Then, there exist $c, \alpha, \beta, \gamma > 0$ s.t. for all $B(x_0, r) \subset \Omega$ the following holds

$$\sup_{B(x_0,\frac{r}{2})} u \leq \inf_{B(x_0,r)} u + c\, r^{\alpha} \left(\int_{B(x_0,r)} |\nabla u|^t \, dx \right)^{\beta} \left(\int_{B(x_0,r)} |\nabla u|^2 \, dx \right)^{\gamma}.$$

Main steps

- L^2-sublevel set estimates with $\|u\|_t$ and $\|\nabla u\|_t$ as remainders;
- quantitative version…constants count;
- Harnack type inequality for sublevel sets in the r.h.s.;
- the set where the the inequality fails has measure zero.

Corollary

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be open, connected, with sufficiently smooth boundary and which enjoys the interior sphere condition. Let x_{max} and x_{min} are respectively a local inner maximum and local inner minimum points for $u \in W^{1,t}(\Omega)$, $t > N$. Then, there exists $C = C(N, \Omega) > 0$ and $h \in \mathbb{N}$ such that

$$u(x_{max}) \leq u(x_{min}) + Ch \left(\int_\Omega |\nabla u|^t \, dx \right)^\beta \left(\int_\Omega |\nabla u|^2 \, dx \right)^\gamma$$

where in particular h depends only on $dist(x_{max}, \partial\Omega)$, $dist(x_{min}, \partial\Omega)$.

Sketch of proof

Let $r > 0$ be such that:

i) for all $x \in B(x_{min}, r) \subset \Omega$ one has $u(x) \geq u(x_{min})$;

ii) $\overline{B(x_{min}, r)} \subset \Omega$;

iii) $\overline{B(x_{max}, r)} \subset \Omega$.

Consider the arc $g : [0,1] \longrightarrow \Omega$ such that $g(0) = x_{min}$ and $g(1) = x_{max}$. Let $t_0 = 0 < \dots t_h = 1$ be a partition of $[0,1]$ such that setting $x_i = g(t_i)$ one has

$$B\left(x_i, \frac{r}{2}\right) \cap B\left(x_{i+1}, \frac{r}{2}\right) \neq \emptyset, \quad i = 0, \dots, h-1$$

and where r is such that $B(x_i, r) \subset \Omega$. By the previous Theorem we have

$$\sup_{B(x_0, \frac{r}{2})} u \leq u(x_{min}) + c\, r^{\alpha} \left(\int_{B(x_0, r)} |\nabla u(x)|^t \, dx \right)^{\beta} \left(\int_{B(x_0, r)} |\nabla u(x)|^2 \, dx \right)^{\gamma},$$

which we rewrite in the following form

$$\forall x \in B\left(x_0, \frac{r}{2}\right), \qquad u(x) \leq \sup_{B(x_0, \frac{r}{2})} u \leq u(x_{min}) + N_0, \tag{0.6}$$

where we set for $i = 0, \dots, h-1$

$$N_i := c(N, \Omega) \left(\int_{B(x_i, r)} |\nabla u(x)|^t \ dx \right)^{\beta} \left(\int_{B(x_i, r)} |\nabla u(x)|^2 \ dx \right)^{\gamma} .$$

Now inequality (0.6) in particular holds for

$$x \in B\left(x_1, \frac{r}{2}\right) \cap B\left(x_0, \frac{r}{2}\right)$$

and thus

$$\inf_{B(x_1, \frac{r}{2})} u \leq u(x) \leq u(x_{min}) + N_0 \ . \tag{0.7}$$

By applying iteratively the above Harnack type inequality we end up with

$$\sup_{B(x_h, \frac{r}{2})} u \ \leq \ u(x_{min}) \ + \ N_h \ + \ \cdots \ + \ N_1 \ + \ N_0 \ .$$

Theorem (C.–Tarsia, 2010/2021)

Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, *be an open connected and bounded set, with sufficiently smooth boundary and which satisfies the interior sphere condition. Let* $u \in W^{2,2}(\Omega)$ *be a weak solution to*

$$\begin{cases} \Delta^2 u - \gamma \Delta u = f, & \text{in } \Omega \subset \mathbb{R}^N,\ \gamma \geq 0 \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\ . \end{cases}$$

where $f \in L^2(\Omega)$, $f \geq 0$ *in* Ω *and* $|\{x : f(x) > 0\}| > 0$. *Then, there exists* $\gamma_0 > 0$ *(which depends on the diameter of* Ω*, Sobolev and Poincaré best constants but does not depend on* f*), such that for* $\gamma > \gamma_0$ *one has* $u > 0$ *in* Ω.

Remarks

- generalises to uniformly elliptic operators of order $2m$;
- $N \geq 4$ j.w.w. C. Polvara (Eichmann-Schätzle '22);
- w.i.p. for parabolic, likely fractional as well as nonlinear operators.

Sketch of proof

In order to apply the above Harnack inequality, we estimate first order derivatives of the solution. This is not a direct consequence of elliptic regularity as we need estimates which are uniform with respect to the parameter γ. Here comes the restriction $N < 4$.

Multiplying $\Delta^2 u - \gamma \Delta u = f$ by u and taking into account $u = \nabla u = 0$ on $\partial\Omega$

$$\int_\Omega |\Delta u(x)|^2 \; dx + \gamma \int_\Omega |\nabla u(x)|^2 \; dx = \int_\Omega f(x)\, u(x) \; dx.$$

Moreover

$$\int_\Omega |\Delta u(x)|^2 \, dx = \sum_{i,j=1}^{n} \int_\Omega |D_{ij} u(x)|^2 \, dx =: \int_\Omega \|D^2 u(x)\|^2 \, dx \, .$$

By Sobolev's embedding, Poincaré inequality and from equation, when $N = 3$ and $t = 6$ we have,

$$\|\nabla u\|_{L^t(\Omega)} \leq \frac{c_S}{d_\Omega} \|\nabla u\|_{L^2(\Omega)} + c_S \|D^2 u\|_{L^2(\Omega)} \leq$$
$$\leq c\|D^2 u\|_{L^2(\Omega)} = c\|\Delta u\|_{L^2(\Omega)} \leq c \left(\int_\Omega f(x)\, u(x)\, dx \right)^{\frac{1}{2}} .$$

Similarly when $N = 2$ and $t \geq 1$ we obtain

$$\|\nabla u\|_{L^t(\Omega)} \leq \frac{c_S}{d_\Omega^{1-\frac{2}{t}}} \|\nabla u\|_{L^2(\Omega)} + c_S d_\Omega^{\frac{2}{t}} \|D^2 u\|_{L^2(\Omega)} \leq$$
$$\leq c d_\Omega^{\frac{2}{t}} \|D^2 u\|_{L^2(\Omega)} = c d_\Omega^{\frac{2}{t}} \|\Delta u\|_{L^2(\Omega)} \leq c \left(\int_\Omega f(x) u(x)\, dx \right)^{\frac{1}{2}} .$$

Lemma

Assume $f(x) = 0$ on $\Omega \setminus \Omega_1$, with Ω_1 s.t. $dist(\partial\Omega_1, \partial\Omega) > 0$. Let be $u \in W_0^{2,2}(\Omega)$ a solution of

$$\Delta^2 u - \gamma \Delta u = f \geq 0, \quad \text{in } \Omega \subset \mathbb{R}^N, \ \gamma \geq 0, \int_\Omega f(x)\ dx > 0.$$

Then,

$$\sup_{\Omega_1} u > 0 \quad \text{and} \quad \int_\Omega |\nabla u(x)|^2\ dx \leq \frac{1}{\gamma} \int_{\Omega_1} f(x)\, u(x)\ dx.$$

As a cosequence

$$u(x_{max}) \leq u(x_{min}) + c(d_{\Omega_1}, N)\ \frac{\left(\int_{\Omega_1} f(x) u(x)\ dx\right)^{\mathbf{b}+\mathbf{c}}}{\gamma^{\mathbf{d}}}$$

where **b**, **c**, $\mathbf{d} > 0$.

Distinguishing the cases $\sup u \geq 1$ and $\sup u < 1$, we have

$$u(x_{max}) \leq u(x_{min}) + c(d_\Omega, N)\ \frac{u(x_{max})}{\gamma^{\mathbf{d}}}\ .$$

If $\gamma > \gamma_0 > 0$ we end up with

$$u(x_{max}) \leq c(d_\Omega, N, \gamma_0) u(x_{min}).$$

The last effort is to remove the restriction of compactly supported data.

i) $\overline{\Omega}_m \subset \Omega_{m+1} \subset \overline{\Omega}_{m+1} \subset \Omega$;

ii) $\cup_{m=1}^{\infty} \Omega_m = \Omega$;

iii) $\{x \in \Omega \,:\, |\{f > 0\}| > 0\} \cap \Omega_1 \neq \Omega_1$;

iv) $dist(\partial\Omega_m, \partial\Omega) \longrightarrow 0$ as $m \to \infty$.

Let χ_m be the characteristic function of Ω_m, we apply the MP just proved to

$$\begin{cases} u_m \in W^{4,2} \cap W_0^{2,2}(\Omega), \\ \Delta^2 u_m(x) - \gamma \Delta u_m(x) = g_m(x), \quad x \in \Omega, \end{cases}$$

where

$$g_m = \frac{1}{S(x)} \frac{\chi_m(x)}{m^2} f(x), \quad S(x) = \sum_{m=1}^{+\infty} \frac{\chi_m(x)}{m^2}.$$

There exists γ_m such that for every $\gamma > \gamma_m$ we have $u_m(x) > 0$ on Ω. Actually uniformity holds and γ_m does non depends on the distance of the maximum point of u_m from the boundary and $\exists \gamma_\infty : \forall m \in \mathbb{N}$ $\gamma_m < \gamma_\infty$. Hence $\forall m \in \mathbb{N}$ and $\forall \gamma > \gamma_\infty \Rightarrow u_m > 0$ in Ω.

Finally

$$\begin{cases} v_m \in W^{4,2} \cap W_0^{2,2}(\Omega), \\ \Delta^2 v_m(x) - \gamma \Delta v_m(x) = f_m(x), \quad x \in \Omega\,, \end{cases}$$

where $f_m = \sum_{i=1}^m g_i$, $v_m = \sum_{i=1}^m u_m$ with $v_m > 0$ in Ω.
Next pass to the limit as $m \to \infty$ to get $f_m \longrightarrow f$ in $L^2(\Omega)$, $v_m \longrightarrow v$ in $W^{4,2}(\Omega)$ where $v > 0$ in Ω by construction and solves

$$\begin{cases} v \in W^{4,2} \cap W_0^{2,2}(\Omega), \\ \Delta^2 v(x) - \gamma \Delta v(x) = f(x), \quad x \in \Omega. \end{cases}$$

We conclude by uniqueness that $v = u > 0$ in Ω.