LIOUVILLE PROPERTIES OF DEGENERATE ELLIPTIC FULLY NONLINEAR EQUATIONS

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based on

M.B. - Annalisa CESARONI, JDE 2016;

M.B - ALESSANDRO GOFFI, Calc. Var. PDE 2019
Math. Ann. 2021
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PLAN :

1. PROBLEM: Liouville properties for SUB or SUPERSOLUTIONS

2. An abstract theorem for fully nonlinear PDEs

3. A STRONG MAXIMUM PRINCIPLE for DEGENERATE equations.

4. Lyapunov functions via HOMOGENEOUS NORMS

5. Some results for the Heisenberg group

6. Some results in the Grushin plane

7. More examples (Carnot groups, ...)

FULLY NONLINEAR 2^{nd} order PDE :

$$(E) \qquad F(x, u, Du, D^2u) = 0 \qquad \text{in } \mathbb{R}^n.$$

Standing assumptions :

$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ continuous & PROPER :

$$F(x, r, p, X) \le F(x, s, p, Y) \qquad \forall\ r \le s,\ X - Y \ge 0;$$

F satisfies a COMPARISON PRINCIPLE in all BOUNDED open sets $\Omega \subseteq \mathbb{R}^n$, i.e., u, v sub & supersol. in Ω

$$u \le v \text{ on } \partial\Omega \implies u \le v \text{ in } \Omega.$$

[Well known under some REGULARITY in x & NONDEGENERACY in either u or D^2u]

PROBLEM. (Liouville property for SEMI-SOLUTIONS)

- $u \in C(\mathbb{R}^n)$ BOUNDED VISCOSITY SUB-SOLUTION of (E)

$$F[u] \le 0 \text{ in } \mathbb{R}^n \overset{?}{\implies} u \equiv \text{constant} \quad ?$$

- same question for SUPER-SOLUTIONS

$$F[u] \ge 0 .$$

The answer is often NO

- $\begin{cases} u \text{ bounded}, \\ -\Delta u \leq 0 \text{ in } \mathbb{R}^n \end{cases} \overset{?}{\Rightarrow} u \text{ const}?$ YES if $n=2$ / NO if $n \geq 3$

$u(x) = \dfrac{-1}{\sqrt{1+|x|^2}}$ if $n=3$, $u(x) = \dfrac{-1}{1+|x|^2}$ if $n \geq 4$

- Grushin Laplacian: $\Delta_G u = u_{xx} + x^2 u_{yy}$ in $\mathbb{R}^2$

Associated homogeneous norm:

$\rho(x,y) = (x^4 + 4y^2)^{1/4} \rightarrow -\Delta_G \frac{1}{\rho} = 0 \quad \forall \rho \neq 0$

$u(x) := \begin{cases} \text{polynomial, if } \rho < 1 \\ 1/\rho \quad \text{if } \rho \geq 1 \end{cases}$ is a bounded, NONCOSTANT subsolution.

- Heisenberg Laplacian: $\begin{cases} X_1 = \partial_x + 2y\,\partial_z \\ X_2 = \partial_y - 2x\,\partial_z \end{cases}$ in $\mathbb{R}^3$

$\Delta_H u \doteq (X_1^2 + X_2^2)u$.

$\rho(x,y,z) := \left((x^2+y^2)^2 + z^2\right)^{1/4}$, $u(x,y,z) = \dfrac{-1}{1+\rho^2}$ bounded, non-constant.

& $-\Delta_H u \geq 0$. In any homog. Carnot group with homog. dim Q, $u = \mp(1+\rho^2)^{1-\frac{Q}{2}}$ solves

$\mp \Delta_G u \leq 0$.

N.B. : sharp contrast with Liouville property for SOLUTIONS : in all previous examples SOLUTIONS in $\mathbb{R}^n$ are CONSTANT.

This follows from HARNACK INEQUALITY for solutions.

(See Bonfiglioli-Lanconelli-Uguzzoni for this in Carnot groups)

In fact, for F as above & UNIFORMLY ELLIPTIC all bounded solutions of $F[u]=0$ in $\mathbb{R}^d$ are constant, see Caffarelli, Cabré book 1995.

Sometimes the answer to Liouville property for sub- or supersolutions is YES.

- ORNSTEIN - UHLENBECK diffusion:

$$-\Delta u + \gamma(x-m)\cdot Du \leq 0 \quad \text{in } \mathbb{R}^n, \ \gamma>0, \ m\in\mathbb{R}^n$$

is associated to an ERGODIC diffusion process: subsolutions are constant $\forall n$.

More generally $-\Delta u - b(x)\cdot Du \leq 0$ & $\lim_{|x|\to\infty} b(x)\cdot x = -\infty$ has Liouville property. "RECURRENCE CONDITION"

See Khasminskii, Pardoux, Veretennikov.., GRIGOR'YAN for connections with diffusion processes on MANIFOLDS.

FIRST ORDER TERM, with "CONFINING VECTOR FIELDS", CAN HELP!

- Also NONLINEARITY can help: PUCCI EXTREMAL OPERATORS, $0<\lambda\leq\Lambda$: $X\in S_n$

$$\mathcal{M}^-(X) = \inf\{-\operatorname{tr}(MX) : M\in S_n,\ \lambda I_d\leq M\leq\Lambda I_d\}$$

e_i eigenvalues of X

$$= -\Lambda\sum_{e_i>0} e_i - \lambda\sum_{e_i<0} e_i$$

$$\mathcal{M}^+(X) = \sup\{\text{same}\} = -\lambda\sum_{e_i>0} e_i - \Lambda\sum_{e_i<0} e_i$$

Cutrì – Leoni AIHP 2000 : all bounded supsolutions of

$\mathcal{M}^+_{\lambda,\Lambda} u \leq 0$ are CONSTANT $\Longleftrightarrow$ $n \leq \frac{\Lambda}{\lambda} + 1$

& this is possible in dim $n > 2$ if $\Lambda > \lambda$.

They also study more general $F(x, D^2u) + h(x)u^p \leq 0$

- I. Capuzzo Dolcetta – Cutrì, Chen – Felmer consider

1^{st} order terms "small at ∞".

Instead, we take LARGE 1^{st} order terms with

$b(x) \cdot x < 0$ "b confining vector field",

+ F NONLINEAR & DEGENERATE ELLIPTIC.

An **ABSTRACT LIOUVILLE THEOREM.** Assume

- SUB-ADDITIVITY : $F[\varphi-\psi] \leq F[\varphi] - F[\psi] \quad \forall \varphi, \psi \in C^2$ & $F[\text{const}] \geq 0$.
- SCALING : $F[\xi \varphi] \leq \varphi(\xi) F[\varphi]$, $\varphi > 0 \ \forall \xi > 0$.
- **STRONG MAXIMUM PRINCIPLE** : $F[u] \leq 0$ in $\mathbb{R}^n$ visco. sense,

$$0 \leq u(x_0) = \max_{\mathbb{R}^n} u =: M \implies u \equiv M \text{ in } \mathbb{R}^n.$$

- **LYAPUNOV FUNCTION** or EXHAUSTION FUNCTION : $\exists\, w \in LSC(\mathbb{R}^n)$, $R > 0$:

$$F[w] \geq 0 \text{ on } |x| > R, \qquad \lim_{|x| \to \infty} w(x) = +\infty$$

- "Khasminski test": $u \in USC(\mathbb{R}^n)$, $F[u] \leq 0$

$$\lim_{|x| \to \infty} \frac{u(x)}{w(x)} \leq 0, \qquad u \geq 0, \qquad [\text{e.g., } u \text{ BOUNDED}].$$

Then $u \equiv$ constant. ∎

Harder assumptions to check:

- SMP
- $\exists$ of Lyapunov function.

THE STRONG MAXIMUM PRINCIPLE. Known cases:

- F UNIFORMLY ELLIPTIC: Caffarelli–Cabré book

$$\mathcal{M}^-(X-Y) \le F(x,r,p,X) - F(x,r,p,Y) \le \mathcal{M}^+(X-Y)$$

- $F = L = \sum_{i=1}^{m} X_i^2$ $\quad X_1, \dots, X_m$ Hörmander vector fields,

i.e. $\operatorname{rk} \mathcal{L}(X_1, \dots, X_m) = n \quad \forall x \in \mathbb{R}^n$: Bony 1969

- F = Bellman operators involving Hörmander fields

$= \sup_\alpha L^\alpha$ or $\inf_\alpha L^\alpha$: M.B–F.DaLio 2001–3.

Need the notion of SUBUNIT VECTOR FIELD.

- Fefferman–Phong 1983: $F = -\operatorname{tr}(A(x) D^2 u)$

$z : \mathbb{R}^d \to \mathbb{R}^d$ is SUBUNIT if $A(x) - (z \otimes z)(x) \ge 0$

i.e., $a_{ij} \xi_i \xi_j \ge |z \cdot \xi|^2 \quad \forall \xi \in \mathbb{R}^n, \forall x.$

- F NONLINEAR: DEF. z SUBUNIT if

$$\sup_{\gamma > 0} F(x, 0, p, I - \gamma\, p \otimes p) > 0 \quad \forall p : z(x) \cdot p \ne 0$$

"F strictly decreasing in the direction of the matrix $z(x) \otimes z(x)$".

nondegenerate

M.B. - A. Goffi 2019

Theorem: $Z(\cdot)$ Lipschitz SUBUNIT VECTOR FIELD,

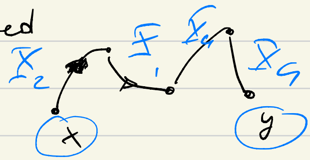
$u \in USC(\Omega)$, $F[u] \le 0$ in Ω, $0 \le u(x_0) = \max_{\Omega} u =: M$,

$\dot{y}(s) = Z(y(s))$, $y(0) = x_0$ $\Longrightarrow$ $u(y(s)) = M \;\; \forall s$.

Corollary 1. $X_1, \dots, X_m$ subunit vector fields:

any $x, y \in \mathbb{R}^n$ can be joined

by concatenating integral



curves of X_i $\Longrightarrow$ SMP holds for $F[u] \le 0$.

Corollary 2 F has $X_1, \dots, X_m$ subunit vector fields satisfying the Hörmander condition.

$\Longrightarrow$ SMP holds.

Main Example: FULLY NONLINEAR SUBELLIPTIC EQUATIONS:

$\mathcal{X} = \{X_1, \dots, X_m\}$ vector fields, $D_{\mathcal{X}} u := (X_1 u, \dots, X_m u)$

$(D^2_{\mathcal{X}} u)_{ij} := X_i(X_j u)$ horizontal gradient & Hessian.

$(D^2_{\mathcal{X}} u)^* =$ symmetrized horizontal Hessian.

(SE) $$G(x, u, D_{\mathcal{X}} u, (D^2_{\mathcal{X}} u)^*) = 0 \quad \text{in } \mathbb{R}^n$$

with G proper.

Assume G uniformly elliptic w.r.t. $(D^2_{\mathcal{X}} u)^*$:

$$\mathcal{M}^-_{\lambda,\Lambda}(M-N) \le G(x,r,p,M) - G(x,r,p,N) \le \mathcal{M}^+_{\lambda,\Lambda}(M-N)$$

$\mathcal{M}^{\pm}_{\lambda,\Lambda}$ Pucci operators on $\mathcal{S}_m$.

Rewrite (SE) in Euclidean coordinates as

$F(x, u, Du, D^2 u) = 0$: the vector fields $X_1, \dots, X_m$ are SUBUNIT for this F.

REDUCTION to PUCCI INEQUALITIES: 1: SUB SOL.

Assume

$$G(x,r,p,O) \geq H_i(x,r,p) := \underbrace{\inf_\alpha \{ c^\alpha(x) r - b^\alpha(x)\cdot p \}}.$$

Then

$$\underline{\mathcal{M}^-_{\lambda,\Lambda}((D^2_x u)^*) + H_i(x,r,D_x u)} \leq G(x,u,D_x u,(D^2_x u)^*) \leq 0$$

$\Rightarrow$ a SUB-solution of (SE) is also a subsolution of $\boxed{\mathcal{M}^- + H_i \leq 0.}$

N.B.: $\mathcal{M}^- + H_i$ is $\underline{\text{1-positively homogeneous and CONVEX}}$

Assume also
- $c^\alpha(x) \geq 0$, c^α loc. equicont.
- b^α loc. uniformly Lip.

Then SMP holds for $\mathcal{M}^-((D^2_x u)^*) + H_i(x,u,D_x u) \leq 0$

& then also for $G(\quad) \leq 0$.

REDUCTION to PUCCI INEQUALITIES : 2- supersols.

Assume

$$G(x,r,p,0) \leq H_J(x,r,p) = \sup_\alpha \{ c^\alpha(x) r - b^\alpha(x)\cdot p \} .$$

Then

$$\mathcal{M}^+_{\lambda,\Lambda}((D^2_X u)^*) + H_J(x,r,D_X u) \geq G(x,r,D_X u, (D^2_X u)^*)$$

$\Rightarrow$ can reduce to SUPER-SOLUTIONS of $\mathcal{M}^+ + H_J$.

1 st. CONCLUSION : the LIOUVILLE property holds for this class of equations = UNIFORMLY SUBELLIPTIC EQUATIONS as soon as THERE EXIST A LYAPUNOV FUNCTION.

How to build LYAPUNOV FUNCTIONS? i.e.

$w \in LSC(\mathbb{R}^n \setminus B_R)$, $F[w] \geq 0$ in $\mathbb{R}^n \setminus B_R$.

- For NON-DEGENERATE operators:

$w(x) = \log|x|$, or .. $w(x) = |x|^2$ .....

- For operators designed on CARNOT GROUPS

or GRUSHIN-TYPE GEOMETRIES:

$\rho(x) =$ HOMOGENEOUS NORM

- Ex. 1 Heisenberg $\mathbb{H}^d = \mathbb{R}^{2d+1}$, $x = (x_1, \dots, x_{2d}, x_{2d+1}) = (x_H, x_{2d+1})$

$$\rho_{\mathbb{H}^d}(x_H, x_{2d+1}) = \left(|x_H|^4 + x_{2d+1}^2\right)^{1/4}$$

e.g., $d = 1$, $\rho_{\mathbb{H}}(x, y, z) := \left((x^2+y^2)^2 + z^2\right)^{1/4}$

- Ex. 2 Grushin plane $\mathbb{R}^2$, 2 vector fields

$X_1 = \partial_x$, $X_2 = x\,\partial_y$. $\rho_G(x,y) := (x^4 + 4y^2)^{1/4}$.

We mostly use $w(x) = \log \rho(x)$.

• <u>Ex. 3</u> $X_1, \dots, X_m$ generators of a homogeneous Carnot group $\mathbb{G} \sim \mathbb{R}^n = \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_r}$, $d_1 = m$.

Dilations, $\lambda > 0$: $\delta_\lambda : \mathbb{R}^d \to \mathbb{R}^d$, $\delta_\lambda(x^{(1)}, \dots, x^{(r)}) = (\lambda x^{(1)}, \dots, \lambda^r x^{(r)})$

HOMOGENEOUS DIMENSION: $Q = \sum_{i=1}^{r} i\, d_i > n$.

Known that [Folland, see Bonfiglioli-Lanconelli-Uguzzoni book]

$\exists$ symmetric norm ρ, HOMOGENEOUS w.r.t. δ_λ,

smooth in $\mathbb{R}^n \setminus \{0\}$, $\Delta_{\mathbb{G}}(\rho^{2-Q}) = 0$ in $\mathbb{R}^n \setminus \{0\}$.

Moreover $\exists\, \gamma > 0$: $\gamma \rho^{2-Q}$ is THE FUNDAMENTAL SOLUTION of $\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i^2$.

In many cases can compute explicitly

$$D_X w =: D_{\mathbb{G}} w \quad \& \quad D_X^2 w =: D_{\mathbb{G}}^2 w$$

for $w = \log \rho(x)$.

Examples: 1. The EFFECT of DIMENSION.

In Heisenberg $\mathbb{H}^d$: $\mathcal{M}^+_{\lambda,\Lambda}\left(\left(D^2_{\mathbb{H}^d} u\right)^*\right) \leq 0$, $u \geq 0$,

$$\limsup_{|x|\to\infty} \frac{u(x)}{\log \rho(x)} \leq 0 \quad \Rightarrow \quad u \text{ const.}$$

Liouv. prop.ty

$$\Longleftrightarrow \quad Q \leq \frac{\Lambda}{\lambda} + 1 \qquad Q = 2d+2 \text{ the}$$

HOMOGENEOUS DIM.

Same condition as in $\mathbb{R}^n$, with n REPLACED by Q.

Example 2: The effect of 1st ORDER TERMS.

NONDEGENERATE EQS.: $\mathcal{X} = \{\partial x_1, \dots, \partial x_n\}$:

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) + \inf_\alpha \{c^\alpha u - b^\alpha \cdot Du\} \le 0, \quad \limsup_{|x|\to\infty} \frac{u(x)}{\log|x|} \le 0,$$

$u \ge 0$,

(C)

$$\sup_\alpha \{b^\alpha \cdot x - c^\alpha |x|^2 \log|x|\} \le \lambda - \Lambda(n-1) \quad (|x| \ge R)$$

"$b^\alpha \cdot x < 0$ enough"

either

$\Rightarrow u \equiv$ constant.

or $c^\alpha \geq c_0 > 0$ & b^α small

N.B. Condition (C) is sharp if

- $b^\alpha \equiv 0$, $c^\alpha \equiv 0$, $\lambda = \Lambda$ $\quad (-\Delta u \le 0)$

- $b^\alpha = b$ $\forall \alpha$, $c^\alpha \equiv 0$, i.e. $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) - b \cdot Du \le 0$

Example 3 UNIFORMLY SUBELLIPTIC EQS. in Heisenberg $\mathbb{H}^d$ with 1st order terms involving $D_{\mathbb{H}} u$:

$$\mathcal{M}^-_{\lambda,\Lambda}\left((D^2_{\mathbb{H}} u)^*\right) + \inf_\alpha \{ c^\alpha u - b^\alpha D_{\mathbb{H}} u \} \leq 0 .$$

For $w = \log \rho_{\mathbb{H}^d}$, can compute $\mathcal{M}^-_{\lambda,\Lambda}\left((D^2_{\mathbb{H}} w)^*\right)$ & get Liouville property if

$$\sup_\alpha \left\{ b^\alpha \cdot D_H \rho \frac{\rho^3}{|x_H|^2} - c^\alpha \frac{\rho^4 \log \rho}{|x_H|^2} \right\} \overset{(C_{\mathbb{H}})}{\leq} \lambda - \Lambda (Q-1)$$

for $|x| \geq R$

RMKs: ▶ $(C_{\mathbb{H}})$ similar to (C), with Q for n, and $\frac{\rho^4}{|x_H|^2}$ in the role of $|x|^2$.

▶ Consistent with the case $-\Delta_{\mathbb{H}} u \leq 0$

▶ sharp for $\mathcal{M}^-_{\lambda,\Lambda}\left((D^2_{\mathbb{H}} u)^*\right) - b \cdot D_{\mathbb{H}} u \leq 0$

SUFFICIENT CONDITIONS for $(C_{\mathbb{H}})$:

EITHER

$$c^{\alpha}(x) \geq c_0 > 0 \quad \& \quad b^{\alpha} \cdot D_{\mathbb{H}} \rho \leq C \rho \,,$$

OR " $b^{\alpha} \cdot D_{H} \rho$ NEGATIVE ENOUGH".

E.G., $d = 1$, $\quad \eta := \rho^3 D_H \rho = \left(x(x^2+y^2)+yz,\, y(x^2+y^2)-xz\right)$

a sufficient condition for $(C_{\mathbb{H}})$ is

$$\limsup_{|X|\to\infty} b(x) \cdot \frac{\eta}{x^2+y^2} < 2 - 3\Lambda \,.$$

EXAMPLE 4. Uniformly subelliptic eqs. on $\mathbb{H}^d$ with 1^{st} order terms involving Du, the Euclidean ∇.

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathbb{H}}u)^*) + \inf_\alpha b^\alpha(x)\cdot Du \le 0 \quad , \quad b^\alpha : \mathbb{R}^n \to \mathbb{R}^n ,$$

has Liouville property if $\exists\ \gamma > 0$:

$$\sup_\alpha b^\alpha \cdot D\rho \le -\gamma\, x\cdot D\rho + o\left(\frac{1}{\rho^3}\right) \quad \text{as } \rho \to \infty .$$

[for the LINEAR case $\lambda = \Lambda$, $\alpha \in$ singleton,

cfr. Mannucci - Marchi - Tchou 2016] .

EXAMPLE 5. Uniformly subelliptic eqs. with GRUSHIN vector fields: $\begin{cases} X_1 = \partial_x \\ X_2 = x\partial_y \end{cases}$ in $\mathbb{R}^2$

$$\rho(x,y) = (x^4 + 4y^2)^{1/4}$$

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) + \inf_\alpha \{c^\alpha u - b^\alpha \cdot D_{\mathcal{X}} u\} \leq 0 \quad \text{in } \mathbb{R}^2$$

has Liouville prop. if, for $\eta = (x^3, 2xy)$

$$2\sup_\alpha \{b^\alpha \cdot \eta - c^\alpha \rho^4 \log \rho\} \leq (-\Lambda - \lambda)x^2 + (\lambda - \Lambda)\sqrt{9x^4 + 4y^2}, \quad |x|, |y| > R.$$

Rmk: Same interpretation as for $\mathbb{H}^d$: either "$b^\alpha \cdot \eta < 0$ enough" or $c^\alpha \geq c_0 > 0$.

- Also in this case the condition is optimal.

OTHER EXAMPLES

- CARNOT GROUPS of STEP 2 $<$ H-type / free

- Generalized GRUSHIN : $\begin{cases} X_i = \partial_{x_i} & \text{in } \mathbb{R}^d \\ Y_j = |x|^\gamma \partial_{y_j} & \text{in } \mathbb{R}^k \end{cases}$

$Q = n + (1+\gamma)k \quad , \gamma > 0.$

$\rho(x,y) = \left(|x|^{2(1+\gamma)} + (1+\gamma^2)|y|^2 \right)^{\frac{1}{2+2\gamma}}$

- Heisenberg – GREINER vector fields

- For general Carnot groups : NON-OPTIMAL sufficient conditions for $-\Delta_G u + H(x,u,D_G u) \le 0$.

Main difficulty : compute $\mathcal{M}^{\pm}_{\lambda,\Lambda}\left((D^2_G \log \rho)^* \right)$.

▶ Further reference : CIRANT – GOFFI 2021 for

$$F(x,D^2u) \geq \begin{cases} u^q + |Du|^\gamma \\ \pm u^q |Du|^\gamma - b(x) \end{cases}.$$

A FEW OTHER RELATED RECENT REFERENCES

- Birindelli - Galise - Leoni 2017
- " " - Ishii 2018 & 2021
- " - Demengel - Leoni 2021
- Ferrari - Vitolo 2020

APPLICATIONS & PROBLEMS:

- long time behaviour of Sols. to PARABOLIC Eqs.

$$u_t + F(x, u, Du, D^2u) = 0$$

[M.B.-Cesaroni].

- Same for DEGENERATE PARABOLIC EQS.:

in progress! $\quad u_t - \Delta_{\mathbb{G}} u + H(x, D_{\mathbb{G}} u) = 0$

- CRITICAL VALUE for fully nonlinear elliptic eqs.: $\exists\, c \in \mathbb{R}$:

$$F(x, u, Du, D^2u) = c \quad \text{in } \mathbb{R}^n$$

has a solution with a given growth at ∞!

(M.B.-Cesaroni UNIF. ELL.; Mannucci-Marchi-Tchou LINEAR in $\mathbb{H}^1$)

- HOMOGENIZATION ....
- REGULARITY ...

THANKS for YOUR ATTENTION!