Blow-up phenomena in a half-space

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Main goal

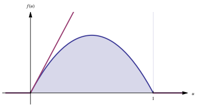
Determine if solutions, starting from $u_0 \geq 0$, to

$$\partial_t u = \mathcal{D}[u] + u^{1+p}, \quad t > 0, x \in \Omega, \quad (p > 0),$$

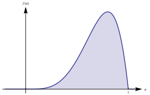
are global in time or not.

- Direct implications on the population dynamics models

$$\partial_t u = \mathcal{D}[u] + u^{1+p}(1-u),$$



vs.



KPP ($p = 0$) vs. Allee effect ($p > 0$)

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The Heat equation

The solution $v(t,x)$ to

$$\partial_t v = \Delta v, \quad v(t=0,\cdot) = v_0,$$

is given by

$$v(t,x) = G(t,\cdot) * v_0(x).$$

It is global and satisfies

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C(v_0)}{(1+t)^{N/2}}.$$

Nonlinear ODE: systematic blow up

For $p > 0$, the solution $u(t)$ to

$$\frac{du}{dt} = u^{1+p}, \quad u(0) = u_0 > 0,$$

always blows up in finite time.

$\partial_t u = \Delta u + u^{1+p}$, $p > 0$

Starting from a nontrivial compactly supported $u_0 \geq 0$, what happens to the Cauchy problem?

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Theorem (Fujita 1966)

Define $p_F := \frac{2}{N}$.

(i) $0 < p \leq p_F \implies$ *all solutions blow up in finite time.*

(ii) $p > p_F \implies$ *some solutions ("small" u_0) are global and get extinct "like the Heat equation".*

Proof for the case $p > p_F$

- Look for a supersolution in the form $u^+(t,x) := g(t)v(t,x)$ where $v := e^{t\Delta}u_0$, $g(0) = 1$, and hope that g is global...

- This requires

$$\frac{g'(t)}{g^{1+p}(t)} \geq v^p(t,x).$$

- It is enough to select

$$\frac{g'(t)}{g^{1+p}(t)} = \left(\frac{C(u_0)}{(1+t)^{N/2}}\right)^p, \quad g(0) = 1,$$

whose solution is computable and global (thanks to $p > \frac{2}{N}$) if "u_0 is small enough".

Proof for the case $p < p_F$

Assume u is global and consider $f(t) := \int_{\mathbb{R}^N} G(t,y)u_0(y)\,dy$.

- Linear argument: $f(t)$ is nothing else than $v(t,0)$ where $v = e^{t\Delta}u_0$ so that

$$f(t) \gtrsim \frac{1}{t^{N/2}}.$$

- Nonlinear argument: $f(t)$ is nothing else than $g(0)$ where

$$g(s) := \int_{\mathbb{R}^N} G(t-s,y)u(s,y)dy.$$

Compute

$$g'(s) = \int -\Delta G u + G\Delta u + Gu^{1+p} = \int Gu^{1+p} \geq \left(\int Gu\right)^{1+p} = g^{1+p}(s).$$

Based on this, find

$$f(t) \lesssim \frac{1}{t^{1/p}}.$$

Nonlocal dispersal (seeds)

$$\partial_t v = J * v - v.$$

J is a probability density on $\mathbb{R}^N$.

$J(x-y)$ is the probability of "jumping" from y to x.

$\int_{\mathbb{R}^N} J(x-y)v(t,y)\,dy = J * v(t,x)$ is the rate at which individuals arrive at x from all other positions.

$-\int_{\mathbb{R}^N} J(y-x)v(t,x)\,dy = -v(t,x)$ is the rate at which individuals leave x to reach any other positions.

Integro differentiel equations $\partial_t u = J * u - u + u^{1+p}$

Starting from a nontrivial compactly supported $u_0 \geq 0$, what happens to the Cauchy problem?

By nonlocal diffusion, individuals are sent in the region $u \approx 0$ where growth is not optimal (Allee effect) so it is more difficult to blow up.

The heavier the tails of J, the smaller should be the Fujita exponent...

On the linear equation $\partial_t v = J * v - v$

Assumption

$$\widehat{J}(\xi) = 1 - A|\xi|^\beta + o(|\xi|^\beta), \text{ as } \xi \to 0,$$

for some $0 < \beta \leq 2$, $A > 0$.

Rk: if J has a second momentum then $\beta = 2$, but for heavier tails $0 < \beta < 2$. In many cases β can be explicitly computed from the tails of J (see stable laws in probability theory).

Theorem (Chasseigne, Chaves and Rossi 2006)

Solutions to $\partial_t v = J * v - v$ *decrease like* $\dfrac{1}{(1+t)^{N/\beta}}$.

$$\partial_t u = J * u - u + u^{1+p},\ p > 0$$

Theorem (A. 2015)

Define $p_F := \frac{\beta}{N}$.

(i) $0 < p \leq p_F \Longrightarrow$ *all solutions blow up in finite time.*

(ii) $p > p_F \Longrightarrow$ *some solutions ("small" u_0) are global and get extinct "like the Heat equation".*

Rk: as expected

$$p_F(NONLOCAL) \leq p_F(LOCAL).$$

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A general framework

We now work in the half-space $\mathbb{R}^N_+$ and consider

$$\begin{cases} \partial_t u = Au + |u|^p u & t > 0, x \in \mathbb{R}^N_+, \\ u(0,x) = u_0(x) \geq 0 & x \in \mathbb{R}^N_+, \\ u(t,x) = 0 & t > 0, x \in \partial\mathbb{R}^N_+. \end{cases}$$

where $Au = \Delta u$ or $Au = -(-\Delta)^{\beta/2} u$ or $Au = J * u - u$.

▶ This Cauchy problem in $\mathbb{R}^N_+$ is understood as the restriction of the Cauchy problem in $\mathbb{R}^N$ obtained by anti-symmetrization.

Fujita critical exponent

Lemma (A., Kavian 2021)

Solutions to $\partial_t v = Av$ *decrease like* $\frac{1}{(1+t)^{(N+1)/\beta}}$.

Theorem (A., Kavian 2021)

$$p_F(\textit{half-space}) = \frac{\beta}{N+1}.$$

Rk: as expected

$$p_F(\text{half-space}) \leq p_F(\text{whole space}).$$

Rk 1: where to evaluate v?

The proof of the blow-up relies on a lower bound "with the good magnitude" of v the solution to the linear diffusion equation.

- In $\mathbb{R}^N$, considering $v(t, 0)$ was enough.
- In $\mathbb{R}^N_+$, we need to evaluate $v(t, \cdot)$ at an appropriate moving point:

Lemma (Pointwise estimate from below)

There exist two constants $\gamma > 0$ and $C = C(\gamma) > 0$ such that

$$v\left(t, \gamma t^{1/\beta} \mathbf{e}_N\right) \geq \frac{C m_1(v_0)}{t^{(N+1)/\beta}}, \quad \forall t \gg 1.$$

Rk 2: role of the boundary conditions

Thus, in the half-space (say for $A = \Delta$):

$$p_F(Dirichlet) = \frac{2}{N+1}.$$

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$$p_F(Robin) = \frac{2}{N+1}.$$

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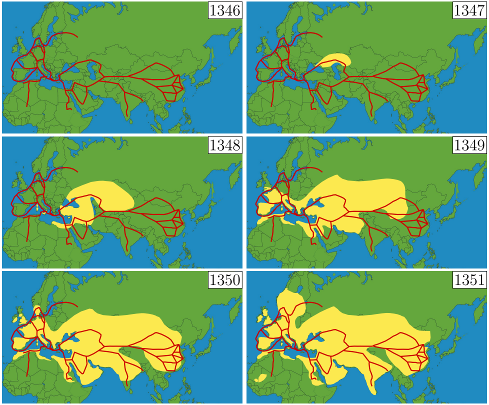
Motivations



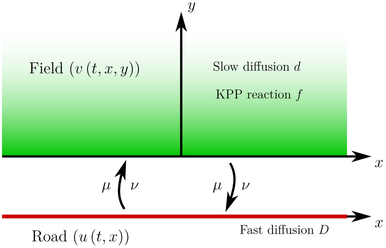
Motivations


Google Earth


Credit photo: Liam Harrap

The road-field model (Berestycki, Roquejoffre and Rossi)

$$\begin{cases} \partial_t v = d\Delta_{x,y} v + f_{KPP}(v), & t > 0,\ x \in \mathbb{R}^{N-1},\ y > 0, \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & t > 0,\ x \in \mathbb{R}^{N-1}, \\ \partial_t u = D\Delta_x u + \nu v|_{y=0} - \mu u, & t > 0,\ x \in \mathbb{R}^{N-1}. \end{cases}$$



- $D > 2d \implies$ acceleration of invasion.

The purely diffusive road-field model

$$\begin{cases} \partial_t v = d\Delta_{x,y} v, & t > 0,\ x \in \mathbb{R}^{N-1},\ y > 0, \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & t > 0,\ x \in \mathbb{R}^{N-1}, \\ \partial_t u = D\Delta_x u + \nu v|_{y=0} - \mu u, & t > 0,\ x \in \mathbb{R}^{N-1}. \end{cases}$$

Movie

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Movie

▶ Fourier (on the road variable)/Laplace (on time) transform is a good strategy and provides:

The solution explicitly

Theorem (A., Ducasse, Tréton 2022)

$$v(t,X) = \mathbb{V}(t,X) + \frac{\mu}{\sqrt{d}} \int_{\mathbb{R}^{N-1}} \Lambda(t,z,y)\, u_0(x-z)\, dz + \frac{\mu\,\nu}{\sqrt{d}} \int_0^t \int_{\mathbb{R}^{N-1}} \Lambda(s,z,y)\, \mathbb{V}|_{y=0}(t-s,x-z)\, dz\, ds,$$

$$u(t,x) = e^{-\mu t}\mathbb{U}(t,x) + \nu \int_0^t e^{-\mu(t-s)} \int_{\mathbb{R}^{N-1}} G_R(t-s,x-z)\, v|_{y=0}(s,z)\, dz\, ds,$$

where $\mathbb{V}$, $\mathbb{U}$, G_R are well-known while Λ, the keystone for writing the solution, is explicit but “not so nice”...

$\Lambda = \Lambda(t, x, y)$ is defined as

$$\frac{e^{-\frac{y^2}{4dt}}}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} \Big[a\alpha\Phi_\alpha + b\beta\Phi_\beta + c\gamma\Phi_\gamma\Big](t, \xi, y)\, e^{-dt\|\xi\|^2 + i\xi\cdot x}\, d\xi,$$

with $(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma)(\xi)$ being the three complex roots of the δ-indexed polynomials

$$P_\delta(\sigma) = \sigma^3 + \frac{\nu}{\sqrt{d}}\sigma^2 + (\mu + \delta)\sigma + \frac{\nu\delta}{\sqrt{d}}, \quad \text{with } \delta = (D - d)\|\xi\|^2,$$

$(a, b, c) = (a, b, c)(\xi)$ being given by

$$a = \frac{1}{(\alpha - \beta)(\alpha - \gamma)}, \quad b = \frac{1}{(\beta - \alpha)(\beta - \gamma)}, \quad c = \frac{1}{(\gamma - \alpha)(\gamma - \beta)},$$

and for $\bullet \in \{\alpha, \beta, \gamma\}$,

$$\Phi_\bullet(t, \xi, y) = \frac{\text{Erfc}}{\Gamma}\left(\frac{-2\bullet\sqrt{d}t + y}{2\sqrt{dt}}\right),$$

where $\Gamma(\ell) = e^{-\ell^2}$, and Erfc is the complementary error function.

Asymptotic decay

Theorem (A., Ducasse, Tréton 2022)

We have

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^N_+)} \lesssim \frac{C_{v_0}\ln(1+t) + C_{u_0,v_0}}{(1+t)^{N/2}}, \qquad \forall t > 0,$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^{N-1})} \lesssim \frac{C_{v_0}\ln(1+t) + C_{u_0,v_0}}{(1+t)^{N/2}}, \qquad \forall t > 0.$$

Fujita blow-up phenomena on the road-field

$$\begin{cases} \partial_t v = d\Delta_{x,y} v + v^{1+p}, & t > 0,\ x \in \mathbb{R}^{N-1},\ y > 0, \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & t > 0,\ x \in \mathbb{R}^{N-1}, \\ \partial_t u = D\Delta_x u + \nu v|_{y=0} - \mu u, & t > 0,\ x \in \mathbb{R}^{N-1}. \end{cases}$$

- Ongoing work by Samuel Tréton...

Thanks for your attention.