Mostly Maximum Principle

IV edition

Abstracts and schedule

Cortona
From 30^{th} May to 3^{rd} June 2022

Sapienza
Università di Roma

Univerità degli Studi
di Salerno

List of abstracts

Blow-up phenomena in a half-space

Matthieu Alfaro

After recalling the classical blow-up result of Fujita in the whole space, we study the joint influence of nonlocal diffusion and of considering only a half-space. Last, we consider related issues on the so-called field-road model.

Liouville properties of degenerate elliptic fully nonlinear equations

Martino Bardi

The talk is based on joint papers with Alessandro Goffi.
We study Liouville properties for viscosity sub- and supersolutions of fully nonlinear degenerate elliptic PDEs in the whole space, namely, that under a suitable bound at infinity from above and, respectively, from below, they must be constant.
The first main assumption is that the operator has a family of generalized subunit vector fields that satisfy the Hormander condition: this implies a strong maximum principle.
A second main assumption is the existence of a supersolution out of a big ball, that explodes at infinity. Therefore for a large class of operators the problem is reduced to finding such a Lyapunov-like function.
We make this construction for various families of vector fields: the generators of the Heisenberg group, of H-type groups, of free Carnot groups, and the Grushin vector fields. All these cases are relevant in sub-Riemannian geometry and have in common the existence of a homogeneous norm.
We give explicit sufficient conditions on the size and sign of the first and zero-th order terms in the equations and discuss their optimality.

The question of uniqueness of steady states for reaction-diffusion equations in general domains

Henry Berestycki

In this talk, I will report on ongoing work with Cole Graham on uniqueness of bounded positive solutions of semi-linear elliptic equations in general unbounded domains. We study a variety of reaction types, boundary conditions, and domains, and we encounter an unexpected wealth of behavior. To frame these results, I will recall some earlier works on qualitative properties of semi-linear elliptic equations in unbounded domains as well as some elements of the theory of generalized principal eigenvalues. I will also mention a host of open problems.

Regularity of stable solutions to semilinear elliptic equations up to dimension 9: quantitative proofs

Xavier Cabré

The regularity of stable solutions to semilinear elliptic PDEs has been studied since the 1970's. I will present a work, in collaboration with Figalli, Ros-Oton, and Serra, which solves an open problem posed by Brezis in the mid-nineties by proving the regularity of stable solutions up to the optimal dimension 9. I will also describe a more recent paper of mine which provides full quantitative proofs of the corresponding interior and boundary regularity results.

Invariant cones for linear elliptic systems with gradient couplings

Italo Capuzzo Dolcetta

I will discuss first the validity of the weak Maximum Principle (**wMP**) for vector functions $u = (u_1, .., u_m)$ satisfying systems of the form

$$(\mathbf{S}) \qquad Au + Cu \geq 0$$

in a bounded open set Ω of R^n where A is a diagonal matrix of linear degenerate second order elliptic operators of the form

$$\mathrm{Tr}(A_i \nabla^2 u_j) + b_j \cdot \nabla u_i \ \ , \ i = 1, \ldots, m$$

and C is a cooperative matrix.
By (**wMP**) I mean that for any solution of system (**S**) the boundary condition $u_i \leq 0$ on $\partial\Omega$ propagates to $u_i \leq 0$ in Ω for $i = 1, ..., m$.

Next, some counterexamples to the validity of (**wMP**) are discussed when couplings occur also in first order partial derivatives of the u_i.
In this more general setting I will show, through a suitable reduction to a nonlinear scalar equation of Bellman type, that some algebraic condition on the structure of gradient couplings and the cooperativity condition on the matrix of zero order couplings guarantee the existence of cones, which are typically different from the negative cone R^n_-, which are invariant for system (**S**) according to the notion introduced by H.F.Weinberger in *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat. (6) **8** (1975), 295–310.

The presentation is mostly based on the papers:

- ICD, A. Vitolo, Weak Maximum Principle for Cooperative Systems: the Degenerate Elliptic Case, *Journal of Convex Analysis* Volume 28 (2021), No. 2, Special Issue dedicated to Umberto Mosco.

- ICD, L. Rossi and A. Vitolo, Invariant cones for linear elliptic systems with gradient coupling, to appear in PAFA, Special Issue dedicated to L. Nirenberg.

Some New Insights on the Maximum Principle for Higher Order Operators

Daniele Cassani

In the higher order case, it is well known how truncation methods in general fail. We prove a new Harnack type inequality demanding for augmented integrability on the function involved in place of being just a solution to a PDE, which usually yields Caccioppoli's inequality and thus the solution belongs to the corresponding De Giorgi class. Indeed, in this context Di Benedetto-Trudinger '84 prove a Harnack type inequality for functions with membership in suitable De Giorgi classes. We drop this assumption though we assume more regularity in terms of integrability which enables us to prove De Giorgi type pointwise level estimates. As a consequence, we prove the strong maximum principle for uniformly elliptic operators of any even order, which do contain lower order derivatives, in sufficiently smooth bounded domains which enjoy the interior sphere condition. This is done by a limiting procedure starting from compactly supported functions and then extending the results to the solutions of higher order PDEs subject to Dirichlet boundary conditions.

Special algebraic structures for integrability by compensation in PDEs

Francesca Da Lio

In the study of systems of PDEs the maximum principle is not in general "operational". Therefore one must look for alternative ways to get the regularity and compactness of weak solutions to such systems. Some of the key steps to overcome the lack of maximum principle type arguments is the discovery of very special algebraic structures enjoying integrability by compensation properties which permit to obtain a gain of the regularity of the solutions. In this talk we are going to describe some recents gain of integrability results related to some elliptic local and nonlocal systems of PDEs in critical regime.

Boundary behaviour of nonlocal minimal surfaces

Serena Dipierro

Surfaces which minimize a nonlocal perimeter functional exhibit quite different behaviors from the ones minimizing the classical perimeter. We will investigate some structural properties of nonlocal minimal surfaces, and in particular we will discuss the "stickiness phenomenon", namely the strong tendency of adhering at the boundary of the reference domain, presenting some recent results.

Compactness and curvature for 3-webs

Lawrence Craig Evans

Regularity results for a free boundary problem governed by a nonstandard growth operator

Fausto Ferrari

We introduce some results concerning the regularity of flat (or Lipschitz) free boundaries in one-phase problems governed by the $p(x)-$Laplace operator. The results are contained in a couple of papers written in collaboration with Claudia Lederman, IMAS- CONICET and University of Buenos Aires (Argentina).

Spreading speeds and spreading sets for reaction-diffusion equations

François Hamel

The talk is about the large time dynamics of bounded solutions of reaction-diffusion equations with unbounded initial support in $\mathbb{R}^N$. I will present a Freidlin-Gärtner type variational formula for the spreading speeds in any direction. This provides a description of the asymptotic shape of the level sets of the solutions at large time. The formula involves notions of bounded and unbounded directions of the initial support. The results hold for a large class of reaction terms and for solutions emanating from initial conditions with general unbounded support. I will also discuss the sharpness of the results and I will list some counterxamples when the assumptions are not all fulfilled.
The talk is based on some joint works with Luca Rossi.

Comparison and regularity results for nonlocal PDE with coercive Hamiltonians

Olivier Ley

I will present some comparison and regularity results for nonlocal PDE of integro-differential type in presence of a coercive Hamiltonian. When it is possible to write the nonlocal operator in Levy-Ito form, we obtain general comparison and Lipschitz regularity results by generalization of classical techniques from the local framework. For nonlocal operator with general space-dependence, it is more difficult and I will give a first comparison result and discuss Holder regularity.
This is a work in collaboration with A. Ciomaga (Paris), M. T. Le (Vienna) and E. Topp (Santiago).

Non coercive evolutive first order mean field games

Paola Mannucci

The Mean Field Games (MFGs) model describes interactions among a very large number of identical agents. Evolutive first order MFGs occurs when the time horizon is finite and the dynamics of the agents are deterministic; they are modelized by a system of two coupled equation: a Hamilton- Jacobi equation and a continuity equation describing respectively the optimal cost of a generic agent and the distribution of the whole population.
I will talk about a joint research project with Y. Achdou, C. Marchi and N. Tchou about some models of MFGs where the Hamiltonian is not coercive in the gradient term because the dynamics of the generic player must fulfill some constraints or fail to be controllable.
First of all I will outline the model where the generic player can move in the whole space but it has some forbidden directions. Afterwards, I will treat the case where the dynamics of the generic agent are controlled by the acceleration.
We study the existence of weak solutions and we relate it with the relaxed equilibria in the Lagrangian setting which are described by a probability measure on the set of optimal trjectories.

On Bernstein type theorems for minimal graphs under Ricci lower bounds

Luciano Mari

In this talk, we study solutions to the minimal hypersurface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0$$

defined on a complete Riemannian manifold M, which describe entire minimal graphs in $M\times\mathbb{R}$. The behaviour of solutions is influenced by the geometry of M, and natural manifolds to study are those with non-negative Ricci curvature, for which one may expect rigidity results similar to those holding in Euclidean space. However, the lack of uniform ellipticity of the mean curvature operator combined with the mild Ricci curvature assumption make the problem analytically challenging. We shall describe splitting and Liouville type theorems both for positive entire minimal graphs and for minimal graphs having at most linear growth. Techniques range from heat equation to potential theoretic arguments. Especially, we focus on a new gradient estimate for solutions, obtained via maximum principles at infinity and a recent duality result for (fully)nonlinear inequalities.

Regularity for Supersolutions to Fully Nonlinear PDEs Under Convexity Assumptions

Diego Moreira

In this talk, we discuss some sharp regularity results related to supersolutions to fully nonlinear equations that enjoy some convexity property. If time allows, some recent and ongoing results will be discussed.
This is joint work with Alessio Figalli (ETH-Zurich) and J. Ederson Braga (UFC) and more recently with Prof. Edgard Pimentel (University of Coimbra, Portugal).

Overdetermined problems and shape optimization in cones

Filomena Pacella

We present some results about the problem of characterizing domains in cones for which a solution of a partially overdetermined problem exists. This is related to the question of characterizing constant mean curvature surfaces with boundary in cones. In recent papers in collaboration with G. Tralli it is proved that if the cone is convex the only domains or surfaces with the above properties are spherical sectors or spherical caps. This should still hold if the cone is almost convex but it is not expected to be true for general nonconvex cones.
Some recent results in collaboration with A. Iacopetti and T. Weth show that these questions are related to the study of the first nontrivial Neumann eigenvalue of the Laplace-Beltrami operator on domains on the unit sphere. In particular, this allows to determine classes of nonconvex cones for which the spherical sectors are not the minimizers for the associated shape optimization problem for the torsional energy functional. Finally, by a concentration-compactness argument, it is proved that minimizers for the shape optimization problem do exist in some cases but are not radial domains. A similar break of symmetry result holds for constant mean curvature surfaces in some nonconvex cones.

Comparison principles by monotonicity duality and fiberegularity

Kevin Payne

We will discuss the validity of the comparison principle in two (often equivalent) formulations. The first concerns generalized subharmonics and superharmonics in a nonlinear potential theory which is determined by a given subequation (constraint set) in the 2-jet bundle. The second concerns viscosity subsolutions and supersolutions for a fully nonlinear (degenerate) elliptic PDE which is determined by a given operator. The first formulation is interesting in its own right, since in many important geometric contexts one has a potential theory but no natural operator. The equivalence of these two formulations is known as the correspondence principle and holds provided that one has a correspondence relation and compatibility. The cor-

respondence principle provides a potential theoretic approach to studying the PDE. This is particularly fruitful in cases where the given operator needs to be restricted to a proper subset of the 2-jet bundle in order to be (degenerate) elliptic and can yield results in terms of nonstandard structural conditions on the operator. The comparison principle is shown to hold provided that there is sufficient monotonicity and regularity and exploits a natural notion of duality.
The results to be presented are in collaboration with M. Cirant (Università di Padova), F.R. Harvey (Rice University) and H.B. Lawson, Jr. (Stony Brook University) and D.F. Redaelli (Università di Padova).

The space of Hardy-weights for quasilinear equations: Maz'ya-type characterization and sufficient conditions for existence of minimizers

Yehuda Pinchover

Let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^N$ be a domain. Let $A := (a_{ij}) \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^{N\times N})$ be a symmetric and locally uniformly positive definite matrix. Set $|\xi|_A^2 := \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j$, $\xi \in \mathbb{R}^N$, and let V be a real valued potential in a certain local Morrey space. We assume that the energy functional

$$Q_{p,A,V}(\phi) := \int_\Omega [|\nabla\phi|_A^p + V|\phi|^p]\mathrm{d}x$$

is nonnegative in $W^{1,p}(\Omega)\cap C_c(\Omega)$.
We introduce a generalized notion of $Q_{p,A,V}$-capacity and characterize the space of all Hardy-weights for the functional $Q_{p,A,V}$, extending Maz'ya's well known characterization of the space of Hardy-weights for the p-Laplacian. In addition, we provide various sufficient conditions on the potential V and the Hardy-weight g such that the best constant of the corresponding variational problem is attained in an appropriate Beppo-Levi space.
This talk is based on joint work with Ujjal Das.

Mean value formulas, maximum principle and Harnack inequality for classical solutions to degenerate Kolmogorov equations

Sergio Polidoro

We prove mean value formulas for classical solutions to uniformly parabolic equations in divergence form and we derive from them a strong maximum principle and a Harnack inequality. We emphasize that our results rely on the classical PDEs theory, and on a fine result due to Dubovickii on the regularity the level sets of functions. We also discuss some generalization of our main results to degenerate

Kolmogorov equations.
This is a joint work in collaboration with Diego Pallara and Emanuele Malagoli.

Coupling and doubling and timely decay

Alessio Porretta

The convergence to equilibrium of Fokker-Planck equations with confining drift is a classical issue, starting with the basic model of the Ornstein-Uhlenbeck process. In this talk we present a new PDE approach to get estimates on the time decay rate, which applies to both local and nonlocal diffusions. This is based on duality arguments and oscillation estimates for transport-diffusion and Hamilton-Jacobi equations.

Large solutions for fractional Hamilton-Jacobi equations

Alexander Quaas

We study the existence of large solutions for nonlocal Dirichlet problems posed on a bounded, smooth domain, associated to fully nonlinear elliptic equations of order $2s$, with $s \in (1/2, 1)$, and a coercive gradient term with subcritical power $0 < p < 2s$. Due to the nonlocal nature of the diffusion, new blow-up phenomena arise within the range $0 < p < 2s$, involving a continuum family of solutions and/or solutions blowing-up to $-\infty$ on the boundary. This is in striking difference with the local case studied by Lasry-Lions for the subquadratic case $1 < p < 2$.
This work is joint with Gonzalo Dávila and Erwin Topp.

Bi-Kolmogorov type operators and weighted Rellich's inequalities

Abdelaziz Rhandi

In this talk we consider the symmetric Kolmogorov operator $L = \Delta + \frac{\nabla \mu}{\mu} \cdot \nabla$ on $L^2(\mathbb{R}^N, d\mu)$, where μ is the density of a probability measure on $\mathbb{R}^N$. Under general conditions on μ we first prove weighted Rellich's inequalities and deduce that the operators L and $-L^2$ with domain $H^2(\mathbb{R}^N, d\mu)$ and $H^4(\mathbb{R}^N, d\mu)$ respectively, generate analytic semigroups of contractions on $L^2(\mathbb{R}^N, d\mu)$. We observe that $d\mu$ is the unique invariant measure for the semigroup generated by $-L^2$ and as a consequence we describe the asymptotic behaviour of such semigroup and obtain some local positivity properties. As an application we study the bi-Ornstein-Uhlenbeck operator and its semigroup on $L^2(\mathbb{R}^N, d\mu)$.
Work in collaboration with D. Addona, F. Gregorio and C. Tacelli.

Periodic homogenization of the principal eigenvalue of second-order elliptic operators

Andrei Rodríguez-Paredes

We investigate homogenization results for the principal eigenvalue problem associated to 1-homogeneous, uniformly elliptic, second-order operators. Under rather general assumptions, we prove that the principal eigenpair associated to an oscillatory operator converges to the eigenpair associated to the effective one. This includes the case of fully nonlinear operators. Rates of convergence for the eigenvalues are provided for linear and nonlinear problems, under additional regularity and convexity assumptions. Finally, for linear problems, we obtain a rate of convergence for suitably normalized eigenfunctions which is linear in terms of the oscillation parameter.
Joint work with E. Topp, Universidad de Santiago de Chile, and G. Dávila, Universidad Técnica Federico Santa María.

Large time dynamics in nonlocal reaction-diffusion equations

Jean-Michel Roquejoffre

The question is the large time evolution of the level sets in a class of models that are currently known as "Fisher-KPP equations", and in which the diffusion is given by an integral operator. As they arise in domains ranging from probability to epidemiology, understanding these models is important.
The level sets will organise themselves as an invasion front whose position will be asymptotically linear in time, corrected by a logarithmic term. This fact was first observed in the large time study of the extremal particles in the branching brownian motion (Bramson, 1983), and susequently extended to more general random walks. We will describe several situation that are modelled by the equations under consideration, and discuss the mechanisms leading to the linear evolution with the logarithmic correction.

Asymptotic Mean Value Formulas for Solutions of General Second-Order Elliptic Equations

Julio Daniel Rossi

In this talk I will describe asymptotic mean value formulas for solutions of second-order elliptic equations.
The approach is very flexible and allows to consider several families of operators obtained as an infimum, a supremum, or a combination of both infimum and supremum, of linear operators. The equations that one can deal with include well-known operators such as Pucci, Monge-Ampere, Issacs, and $k-$Hessian operators.
Joint work with P. Blanc, F. Charro and J. J. Manfredi.

Are solutions of reaction-diffusion equations asymptotically 1D?

Luca Rossi

The symmetry of solutions of elliptic equations is a classical and challenging problem in PDEs, strictly linked with stability. We consider in this talk parabolic equations and we ask whether the 1-dimensional symmetry eventually emerges in the long time, for solutions which are initially non-symmetric. We will present a satisfactory answer in the case of the Fisher-KPP equation, together with some counter-examples and open questions.
This topic is the object of a joint work with F. Hamel.

A mixed eigenvalue problem on domains tending to infinity in several directions

Itai Shafrir

We analyze the asymptotic behavior of the eigenvalues of elliptic operators in divergence form with mixed boundary conditions, on domains that become unbounded in some of the directions (cylindrical domains).
This is a joint work with Prosenjit Roy.

Overdetermined elliptic problems in the sphere

Pieralberto Sicbaldi

Consider the general Serrin's overdetermined problem in a simply connected subdomain of the sphere. By a very well known result by Kumaresan and Prajapat, if the subdomain is contained in a hemi-sphere then it must be a geodesic ball. The proof of such result generalizes the classical moving plane technique to the sphere. In this talk we will try to understand what happens in the general case, when the subdomain is not contained in any hemi-sphere. This is based on a joint work with D. Ruiz and J. Wu.

Basic elliptic estimates with optimized constants and applications to the qualitative theory of elliptic PDE

Boyan Sirakov

We develop a new, unified approach to the following two classical questions on elliptic PDE: (i) the strong maximum principle for equations with non-Lipschitz nonlinearities, and (ii) the at most exponential decay of solutions in the whole space or exterior domains (Landis conjecture). Our results apply to divergence and nondivergence operators with locally unbounded lower-order coefficients, in a

number of situations where all previous results required bounded ingredients. Our approach, which allows for relatively simple and short proofs, is based on a (weak) Harnack inequality with optimal dependence of the constants in the lower-order terms of the equation and the size of the domain, which we establish. If time permits, we will report on some recent C1 estimates with optimized constants and refined Landis-type results. These are based on a new boundary weak Harnack inequality which also has applications in the boundary regularity theory of equations in divergence form.

Nematic Liquid crystal flows with free boundary

Yannick Sire

I will introduce a new parabolic system for the flow of nematic liquid crystals, enjoying a free boundary condition. After recent works related to the construction of blow-up solutions for several critical parabolic problems (such as the Fujita equation, the heat flow of harmonic maps, liquid crystals without free boundary, etc...), I will construct a physically relevant weak solution blowing-up in finite time. We make use of the so-called inner/outer parabolic gluing.
Along the way, I will present a set of optimal estimates for the Stokes operator with Navier slip boundary conditions. I will state several open problems related to the partial regularity of the system under consideration.
This is joint work with F.-H. Lin (NYU), Y. Zhou (JHU) and J. Wei (UBC).

Diffusive Hamilton-Jacobi equations and their singularities

Philippe Souplet

We consider the diffusive Hamilton-Jacobi equation

$$u_t - \Delta u = |\nabla u|^p$$

with homogeneous Dirichlet boundary conditions, which plays an important role in stochastic optimal control theory and in certain models of surface growth (KPZ). Despite its simplicity, in the superquadratic case $p > 2$ it displays a variety of interesting and surprising behaviors.
We will discuss two classes of phenomena:

- Gradient blow-up (GBU) on the boundary: time rate, single-point GBU, space and time-space profiles, Liouville type theorems and their applications;
- Continuation after GBU as a global viscosity solution with loss and recovery of boundary conditions.

In particular, we will present the recently obtained, complete classification of solutions in one space dimension, which describes the losses and recoveries of boundary conditions at multiple times, as well as all the possible GBU and recovery rates.

This talk is based on a series of joint works in collaboration with A. Attouchi, R. Filippucci, Y. Li, N. Mizoguchi, A. Porretta, P. Pucci, Q. Zhang.

Aleksandrov-Bakelman-Pucci maximum principle for L^n-viscosity solutions of equations with unbounded terms

Andrzej Swiech

We will show that a classical version of the Aleksandrov-Bakelman-Pucci (ABP) maximum principle, in which norms are taken over contact sets, is also true for L^n-viscosity sub/super-solutions of fully nonlinear uniformly elliptic equations with measurable and unbounded terms. More precisely, in this version, the structural coefficient function (corresponding to the drift coefficient for linear equations) belongs to L^n.
Such result was previously known for L^n-viscosity solutions only when this coefficient function was bounded. We will discuss how this version of the ABP maximum principle can be used to extend the theory of L^p-viscosity solutions for such equations and prove their various pointwise properties.
This is a joint work with S. Koike.

Periodic homogenization results for nonlocal Hamilton-Jacobi equations

Erwin Topp

In this talk I will report some results on periodic homogenization for fractional Hamilton-Jacobi equations. We analyze the structure of the problem in terms of the relation among the order of the diffusion and the gradient. We provide positive answers in various regimes for time-dependent equations. Rates of convergence in the case of stationary equations where the diffusion rules the regularity are also presented.

A symmetry result in a free boundary problem

Cristina Trombetti

We study a shape optimization problem involving a solid $K \subset \mathbb{R}^n$ which has constant temperature and it is surrounded by a layer of insulating material Ω which obeys a generalized boundary heat transfer law. We minimize the energy of such configurations among all (K, Ω) with prescribed measure for K and Ω, and without topological or geometrical constraints. In the convection case (corresponding to Robin boundary conditions on $\partial\Omega$) we obtain a full description of minimizers. In the general case, we prove the existence and regularity of solutions and we give a partial description of minimizers.

Schedule

Time	Monday	Tuesday	Wednesday	Thursday	Friday
9:30			Cabré	Dipierro*	Roquejoffre
10:00		Capuzzo Dolcetta	Rhandi	Hamel	Rodríguez-Paredes
10:30		Moreira	Sicbaldi	L. Rossi	Cassani
11:00		Coffee break			
11:30		Sirakov	Polidoro	Quaas	Porretta
12:00	Berestycki	Alfaro	Trombetti	J. Rossi	Pacella
12:30	Pinchover	Payne	*Tribute to Louis Nirenberg*	Evans	
13:00	Lunch				
15:00	Bardi	Da Lio		Shafrir	
15:30	Mannucci	Ley		Topp	
16:00	Coffee break			Coffee break	
16:30	Sire	Souplet		Mari	
17:00	Ferrari	Swiech*			

* Online talks.