Sapienza, Università di Roma (Italy)
Dipartimento di Matematica "Guido Castelnuovo"

Nonlinear Diffusion Problems

dedicated to Maria Assunta Pozio
September 11th–13rd, 2019

Titles & Abstracts (`September 4, 2019`)

Catherine Bandle
University of Basel, Switzerland
Positive solutions of semilinear elliptic problems with a Hardy potential

It is well-known that problems of the form $\Delta u = u^p$, $p > 1$ in a bounded smooth domain D lead to solutions which blow up at the boundary. The first term of their asymptotic behavior is independent of the geometry and can be computed by means of the solution of the one-dimensional problem. In this talk we study the problem $\Delta u + \frac{\mu u}{\delta^2} = u^p$ where δ is the distance to the boundary and μ is a real number. The interplay between the singular potential and the nonlinearity leads to interesting structures of the solutions blowing up at the boundary. The presentation is based on the last paper with Assunta which has been accepted for publication a few day after her death.

Henri Berestycki
PSL University Paris & CNRS, France
Propagation of bistable fronts through a perforated wall: Part II

This is a continuation of the lecture by Hiroshi Matano in which I will describe some proofs. Among other things, I will give an outline of the proof of the De Giorgi type lemma, which plays a key role in establishing the dichotomy theorem mentioned in Matano's lecture. This lemma is stated in a rather general framework and is of independent interest. I will also discuss other related results about the effects of the geometry of the domain on the propagation of bistable fronts.

Elena Beretta
Politecnico di Milano, Italy & New York University Abu Dhabi, UAE
On some inverse boundary value problems related to the monodomain model of cardiac electrophysiology

Ischemic heart disease results from a restriction in blood supply to the heart and represents the most widespread heart disease. As a consequence, myocardial infarction caused by the lack of oxygen might lead to even more severe heart muscle damages, ventricular arrhythmia and fibrillation. Detecting ischemic regions at early stages of their development from noninvasive (or minimally invasive) measurements is thus of primary importance. This is usually performed by recording the electrical activity of the heart, by means of either body surface or intracardiac measurements.

Mathematical and numerical models of the cardiac electrophysiology can be used to shed light on the potentialities of electrical measurements in detecting ischemias. More specifically, the goal is to combine boundary measurements of (body-surface or intracavitary) potentials and a mathematical description of the electrical activity of the heart in order to identify the position, the shape and the size of heart ischemias. The cardiac electrical activity can be comprehensively described in terms of the monodomain model, consisting of a boundary value problem for a semilinear reaction-diffusion equation coupled with nonlinear ordinary differential equations.

In my talk, I will analyze the case of an insulated heart neglecting the coupling with the torso. This results in the challenging inverse problem of detecting conductivity inclusions for the monodomain system with a single measurement of the boundary potential. I will first analyze the steady-state version of the monodomain model since it already exhibits the main features and difficulties of the time-dependent model. Then, I will go through some recent results obtained in the time dependent case.

Michiel Bertsch
Università di Roma Tor Vergata & IAC-CNR, Italy

Discontinuous viscosity solutions of first order Hamilton–Jacobi equations

I consider a Hamilton Jacobi equation in 1 spatial dimension,

$$u_t + H(x, t, u_x) = 0,$$

with initial function $u(x, 0) = u_0(x)$ for $x \in \mathbb{R}$. The hamiltonian $H(x, t, p)$ is a smooth function with at most linear growth in p.

In this talk I shall focus on the issue of uniqueness of discontinuous viscosity solutions. In the 80's Ishii introduced semicontinuous viscosity sub- and supersolutions which satisfy a Comparison Theorem. Unfortunately the Comparison Theorem does not imply uniqueness of viscosity solutions if u_0 is not continuous. In this talk I present a class of equations for which uniqueness of suitably defined discontinuous viscosity solutions can be proved. The proof is based on the "barrier effect" of spatial discontinuities. In addition I comment on counterexamples of uniqueness, which were observed in the 90's by Barles e.a. Some open problems will also be presented.

This talk is based on joint work with F. Smarrazzo, A. Tesei and A. Terracina.

Thierry Cazenave
Sorbonne Université, Paris, France
Some stability properties for minimal solutions of $-\Delta u = \lambda g(u)$

I will report on a joint work with Assunta and M. Escobedo. We study the stability of the branch of minimal solutions $(u_\lambda)_{0<\lambda<\lambda^\star}$ of the equation

$$-\Delta u = \lambda g(u)$$

on a bounded domain of $\mathbb{R}^N$ with Dirichlet boundary conditions, for a nonlinearity g which is neither concave nor convex. We show that stability is related to the regularity of the map $\lambda \mapsto u_\lambda$. We then show that in dimensions $N = 1$ and $N = 2$, discontinuities in the branch of minimal solutions can be produced by arbitrarilly small perturbations of the nonlinearity g. In dimensions $N \geq 3$ the perturbation must be sufficiently large. We also study in detail a specific one-dimensional example.

Jesus Ildefonso Diaz
Universidad Complutense de Madrid, Spain
Beyond the unique continuation: "flat solutions" for reactive slow diffusions and the infinite and confinement singular potentials for the Schrödinger equation

Solutions with compact support for some nonlinear elliptic and parabolic equations, and many other free boundary problems, are formulations for which the Unique Continuation Principle, in its several versions, fails. In such problems, which have attracted the attention of many specialists, and among them Maria Assunta Pozio, the solution u and its normal derivative vanish on a region of the boundary (which leads to the definition of a "flat solution" of the corresponding equation).

In this talk I will present, in a very sketched way, some recent results in this direction, trying to show how many open problems of this nature still remain as a source of current research.

More specifically, I will report on some results concerning the following problems:

I) Stable flat solutions of

$$u_t - \Delta u^m + u^a = \lambda u^b$$

for $0 < a < b < m$ under the stability condition $2(m+a)(m+b) - N(m-a)(m-b) < 0$ (joint work with J. Hernández and Y. Sh. Ilyasov),

II) Flat solutions to

$$\mathbf{i}\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi$$

in $\mathbb{R}^N$, for $V(x) \geq Cd(x, \partial\Omega)^{-\alpha}$, with $\alpha \geq 2$, for some bounded domain Ω (my research continued in collaboration with J. M. Rakotoson, D. Gómez-Castro, R. Temam and J. L. Vázquez).

Jesús Hernández
Universidad Complutense de Madrid, Spain
Positive solutions for a nonlinear elliptic problem arising in chemical reactions

We give an overview on some current work together with V. Bobkov and P. Drábek. We study the the nonlinear elliptic problem with non-Lipschitz nonlinearities

$$\begin{cases} -\Delta u = (1-u)u^m - \mu u^n & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain, $0 < m \leq 1$, $0 < n$ and μ is a real parameter. When $\mu > 0$, this is a model for an isothermal autocatalytic chemical reaction with termination (Leach–Needham).

We obtain different results concerning existence of positive solutions depending on the parameters m, n, and μ by using either sub- and supersolutions or variational arguments, in particular the method of the Nehari manifold. In some cases it is possible to show the linearized stability of the unique positive solution.

Mimmo Iannelli
Università di Trento, Italy
Vector-borne epidemics through a plantation

The model we present is rather general and basic, but it is inspired by the Xylella invasion of olive trees in Apulia, and by the infection of vineyards named Flavescence Dorée in France. These epidemics have common features because are both due to a bacterial agent (*Xylella fastidiosa* for olives and *Candidatus Phytoplasma vitis* for grapevines) vectored by insects (leafhoppers such as *Philaenus spumarius* and *Scaphodeus titanus* respectively).

To describe the dynamics of the infection through the plantation, we will be concerned with a SIR type epidemic model (structured in space), with a force of infection designed to include the vector growth dynamics (structured in age). Indeed vectors are infected by the infected plants and, diffusing through the plantation, transfer the infection to the healthy ones.

We are interested to analyze the final size of the epidemic and to describe the progression of the infection through the plantation.

Robert Kersner
University of Pécs, Hungary
Periodic stationary solutions of some reaction-diffusion systems

I'll consider two reaction-diffusion systems.

The first one is a model for a chemical reaction (Gray-Scott) with linear diffusion. The two families of stationary solutions are related to Jacobi and Weierstrass elliptic functions $\mathrm{sn}(x;k)$ and $P(x;g2;g3)$. For some limit cases of k, g_i one obtains "pulse" and "kink" solutions of Hale–Peletier–Troy.

The second one is a version of Shigesada et al. model with competition-type reaction terms. In dependence of cross-diffusion terms two families of periodic solutions were obtained in terms of circular functions $\sin x$ and $\cos x$. In the first one the solutions have at the same points maxima (and minima); in the second one the points of maxima of one solution coincide with minima of another one.

For all solution we indicate the domains in the parameter space where they exist and show (numerically) that they attract a rather large class of initial functions (Neumann problem).

Joint work with Mihály Klincsik.

Pierangelo Marcati
GSSI - Gran Sasso Science Institute, L'Aquila, Italy

Splash singularities for 2-D incompressible viscoelastic fluids of Oldroyd-B type

Splash singularities are obtained when there exists a time t_*, $0 \leq t_* \leq T$ such that the interface $\partial\Omega(t_*)$ self-intersects in one point. The notion of these singularities has been introduced in [1] and the existence has been proved in the case of viscous incompressible fluids in [2] (see also the alternative approach [3]). In [5], [6] we show the existence of splash singularity for 2-D viscoelastic fluids either in the of case infinite Weissenberg number [5] and in the case of finite Weissenberg [6]. The main idea is to tranform the problem via a conformal transformation which "separate" the singularity, then for the fluid part to follow the Lagrangian dynamics of the linearized problem by using ideas of [4] and the ODE generated by the viscous tensor. Hence a local existence theorem, a stability result and suitably prepared initial data, allow to construct solutions having a splash singularity. In the finite Weissenberg case we also show the dependence of the splash time from the Weissenberg number.

Joint work with E. Di Iorio e S. Spirito.

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[2] A.Castro, D. Cordoba, C. Fefferman, F. Gancedo, J.Gomez-Serrano, Splash singularities for the Free Boundary Navier-Stokes Equations Ann. PDE (2019) 5: 12.
[3] D.Coutand, S.Shkoller On the splash singularity for the free-surface of a Navier–Stokes fluid, Ann. I. H. Poincaré C, Analyse non linéaire 36 (2019), 475–503
[4] T.J.Beale, The initial value problem for the Navier–Stokes equations with a free surface, Commun. Pure Appl. Math. 34 (1981), 359–392.
[5] E.Di Iorio, P.Marcati, S.Spirito, Splash singularity for a free-boundary incompressible viscoelastic fluid model, arXiv:1806.11089
[6] E.Di Iorio, P.Marcati, S.Spirito, Splash singularity for a general Oldroyd model with finite Weissenberg number, arXiv:1806.11136

Hiroshi Matano
Meiji University, Tokyo, Japan
Propagation of bistable fronts through a perforated wall: Part I

We consider a bistable reaction-diffusion equation on $\mathbf{R}^N$ in the presence of an obstacle K, which can be regarded as a wall of infinite span with periodically arrayed holes. More precisely, K is a closed subset of $\mathbf{R}^N$ with smooth boundary such that its projection onto the x_1-axis is bounded, while it is periodic in the rest of variables $(x_2, \ldots, x_N)$. We assume that $\mathbf{R}^N \setminus K$ is connected. Our goal is to study what happens when a planar traveling front coming from $x_1 = +\infty$ meets the wall K.

We first show that there is clear dichotomy between 'propagation' (or invasion) and 'blocking', and that there is no intermediate behavior. This dichotomy result is proved by what we call a De Giorgi type lemma for an elliptic equation on $\mathbf{R}^N$.

Next we discuss sufficient conditions for blocking, and those for propagation. Roughly speaking, blocking occurs if the holes are small. As for propagation, we present three types of walls that allow the front to propagate, namely (a) walls with large holes; (b) small-capacity wall; (c) skeleton walls.

This is joint work with Henri Berestycki and François Hamel.

Masayasu Mimura
Hiroshima University, Japan
Nonlinear diffusion modelling stay (grow)-or-migrate dichotomy

In this talk, we report two nonlinear diffusion models arising in the Neolithic transition in Archaeology and glioma invasion in medical science, respectively. The first is derived by the dispersal of early farmers in the Neolithic transition. The farmers have basically a sedentary life style but if the density of the farmers becomes high, they disperse randomly, that is, this situation implies stay-or-migrate dichotomy. Second is glioma invasion. Experimental evidence suggests that cell motility and proliferation are inversely correlated in gliomas with proliferating tumour cells moving slowly and rapidly migrating tumour cells proliferating slowly. This implies grow-or-migrate dichotomy.

In this talk, we apply the singular limit procedure to derive suitably nonlinear diffusion models which include these dichotomies.

Maria Michaela Porzio
Sapienza, Università di Roma, Italy
Time behavior of the solutions to a class of diffusion problems

We will show some interesting properties of a class to nonlinear parabolic equations appearing in many physical applications, being a nonlinear version of the heat equation. We will describe some regularity properties of these diffusion problems together with the behavior in time of the solutions, with special attention to the autonomous case when the influence of the solutions to suitable elliptic problems appears.

Fabio Punzo
Politecnico di Milano, Italy
The Poisson equation on Riemannian manifolds

The talk is concerned with the existence of solutions to the Poisson equation on complete non-compact Riemannian manifolds. In particular, the interplay between the Ricci curvature and the behavior of the source function will be discussed. This is a joint work with G. Catino and D.D. Monticelli.

Gabriella Tarantello
Università di Roma Tor Vergata, Italy
Minimal immersions of closed surfaces in hyperbolic 3- manifold

The work of K. Uhlenbeck motivated the interest to minimal immersions of closed surfaces of genus larger than 1 into hyperbolic 3-manifolds with prescribed data in the cotangent bundle of the Teichmuller space of the surface. We establish both area minimizing and unstable minimal immersions by proving multiple existence for the Gauss–Codazzi equations. Furthermore, we describe their asymptotic behaviour in terms of a given parameter, and show that the (hyperbolic) metric induced by the immersion acquires conical singularities in the unstable case.
