Self-similar blowup for supercritical wave maps

Roland Donninger (Universität Wien)

Rome, September 12, 2017

Joint work with P. Biernat (Bonn) and B. Schörkhuber (Vienna)

Unterstützt von / Supported by

Alexander von Humboldt
Stiftung / Foundation

From harmonic functions to wave maps

- Laplace's equation

$$\Delta U(x) = \sum_{j=1}^{d} \frac{\partial^2}{\partial(x^j)^2} U(x) = \partial^j \partial_j U(x) = 0$$

for $U : \mathbb{R}^d \to \mathbb{R}$, $x = (x^1, x^2, \ldots, x^d)$, plays fundamental role in mathematics, physics, ...

- Solutions are called *harmonic functions*
- Variational formulation via action functional

$$S(U) = \int_{\mathbb{R}^d} \partial^j U \partial_j U$$

- Laplace's equation is Euler-Lagrange equation associated to S, i.e., formally follows from requirement $\frac{d}{d\epsilon} S(U + \epsilon\varphi)|_{\epsilon=0} = 0$ for all test functions φ

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- Simple generalization to vector-valued $U : \mathbb{R}^d \to \mathbb{R}^n$ by using action

$$S(U) = \int_{\mathbb{R}^d} \partial^j U^a \partial_j U_a$$

- More geometric interpretation:

$$S(U) = \int_{\mathbb{R}^d} \delta^{jk} \partial_j U^a \partial_k U^b \delta_{ab},$$

  where δ is natural Riemannian metric of Euclidean space
- Natural generalization for maps $U : (M, g) \to (N, h)$ between Riemannian manifolds:

$$S(U) = \int_M g^{jk} \partial_j U^a \partial_k U^b h_{ab} \circ U$$

- Solutions of associated Euler-Lagrange equation (nonlinear!) are called *harmonic maps*

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- Lorentzian base manifold, e.g. Minkowski space $\mathbb{R}^{1,d}$ with metric $\eta = \mathrm{diag}(-1, 1, \dots, 1)$, $U : \mathbb{R}^{1,d} \to \mathbb{R}$, action

$$S(U) = \int_{\mathbb{R}^{1,d}} \eta^{\mu\nu} \partial_\mu U \partial_\nu U = \int_{\mathbb{R}^{1,d}} \partial^\mu U \partial_\mu U$$

- Euler-Lagrange equation

$$\partial^\mu \partial_\mu U = -\partial_0^2 U + \partial^j \partial_j U = 0$$

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- Generalization to manifold-valued maps $U : \mathbb{R}^{1,d} \to (M, g)$ yields *wave maps action*

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Why do we care?

- Wave maps action is rich source for *nonlinear geometric relativistic field theories*
- Typical features:
  - Finite speed of propagation
  - Lorentz covariance
  - Dispersion
  - Null structure
- Wave maps occur in physics e.g. as models for ferromagnetism, sigma models in particle physics, toy models for Einstein's equation, etc.
- Wave maps establish link between differential geometry and dispersive PDEs
- We are mathematicians, we don't have to justify our interest in an interesting mathematical object

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Coordinates

- The wave maps equation is complicated!
- Fix $d = 3$ and $M = \mathbb{S}^3$, choose hyperspherical coordinates (ψ, Θ, Φ) on $\mathbb{S}^3$, i.e., $\mathbb{S}^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$ is parametrized by

$$\begin{pmatrix} \psi \\ \Theta \\ \Phi \end{pmatrix} \mapsto \begin{pmatrix} \sin\psi \sin\Theta \sin\Phi \\ \sin\psi \sin\Theta \cos\Phi \\ \sin\psi \cos\Theta \\ \cos\psi \end{pmatrix}$$

  where $\psi, \Theta \in [0, \pi)$ and $\Phi \in [0, 2\pi)$
- Choose spherical coordinates (t, r, θ, φ) on Minkowski space, i.e.,

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ r\sin\theta \sin\varphi \\ r\sin\theta\cos\varphi \\ r\cos\theta \end{pmatrix}$$

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Corotational wave maps

- Map $U : \mathbb{R}^{1,3} \to \mathbb{S}^3$ can be expressed as

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The Cauchy problem

- Eq. (1) is a semilinear wave equation, i.e., study the Cauchy problem

$$\begin{cases} (\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r)\psi(t,r) + \frac{\sin(2\psi(t,r))}{r^2} = 0 \\ \psi(0,r) = f(r), \qquad \partial_0\psi(0,r) = g(r) \end{cases}$$

  for prescribed *initial data* (f,g)
- Questions:
  - Existence, uniqueness, continuous dependence for small times (local well-posedness, LWP) for smooth data
  - LWP for rough data
  - Existence for all times (global well-posedness, GWP), singularity formation

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Finite-time blowup

- Observation: Smooth data do not necessarily lead to smooth solutions for all times:

$$\psi_T(t,r) = 2\arctan\left(\frac{r}{T-t}\right)$$

  is an explicit *self-similar* solution [Shatah 1988, Turok-Spergel 1990]
- Eq. (1) has many self-similar solutions $\psi(t,r) = f_n(\frac{r}{T-t})$ [Bizoń 2000]
- How typical is this? Are these examples just accidents?
- No! Solution ψ_T is conjectured to provide the *generic* blowup profile [Bizoń-Chmaj-Tabor 2000]
- Goal: Develop mathematical understanding of this phenomenon

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Stability of blowup

- If ψ_T plays a role in generic evolutions, it better be stable against perturbations
- What does stability of a blowup solution mean?
- First attempt: ψ_T is stable if perturbed initial data

$$\psi(0, \cdot) = \psi_T(0, \cdot) + f, \qquad \partial_0\psi(0, \cdot) = \partial_0\psi_T(0, \cdot) + g$$

  with (f, g) small lead to solution of the form $\psi(t, r) = \psi_T(t, r)[1 + \varphi(t, r)]$, where $\varphi(t, r) \to 0$ as $t \to T-$ in a suitable sense
- Too naive, perturbation will in general change blowup time T!
- Thus, ψ_T stable if solution is of the form $\psi_{T'}(t, r)[1 + \varphi(t, r)]$ for some $T' \approx T$

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Formulation as a standard semilinear wave equation

- What are the right function spaces to study stability of blowup?
- Singularity at $r = 0$ in

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\psi(t,r) + \frac{\sin(2\psi(t,r))}{r^2} = 0$$

  enforces boundary condition $\psi(t,0) = 0$ for all t
- Natural to switch to $\widehat{u}(t,r) = \frac{\psi(t,r)}{r}$ which satisfies

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right)\widehat{u}(t,r) + \frac{\sin(2r\widehat{u}(t,r)) - 2r\widehat{u}(t,r)}{r^3} = 0$$

- $u(t,x) = \widehat{u}(t,|x|)$ satisfies

$$(\partial_t^2 - \Delta_x)u(t,x) = F(u(t,x),x)$$

  in 5 spatial dimensions and with *smooth* nonlinearity F

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Function spaces

- Blowup solution given by

$$u_T(t,x) = \frac{2}{|x|} \arctan\left(\frac{|x|}{T-t}\right)$$

- Solution does not blow up in energy space:

$$\|u_T(t,\cdot)\|_{\dot{H}^1(\mathbb{B}^5_{T-t})} \simeq (T-t)^{\frac{5}{2}-2}$$

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Stability of blowup in backward lightcone

Theorem (D. 2011, D.-Schörkhuber-Aichelburg 2012, Costin-D.-Xia 2016, Costin-D.-Glogić 2017)

(f,g) *small in* $H^2 \times H^1(\mathbb{B}^5_{3/2})$. *Then the Cauchy problem*

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t,x) = F(u(t,x),x) \\ u(0,x) = u_1(0,x) + f(x), \qquad \partial_0 u(0,x) = \partial_0 u_1(0,x) + g(x) \end{cases}$$

has solution u *in* $\mathcal{C}_T$ *that blows up at* $(t,x) = (T,0)$ *for some* $T \approx 1$ *and*

$$\frac{\|u(t,\cdot) - u_T(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^5_{T-t})}}{\|u_T(t,\cdot)\|_{\dot{H}^k(\mathbb{B}^5_{T-t})}} \lesssim (T-t)^\epsilon, \quad k \in \{0,1,2\}$$

$$\frac{\|\partial_t u(t,\cdot) - \partial_t u_T(t,\cdot)\|_{\dot{H}^\ell(\mathbb{B}^5_{T-t})}}{\|\partial_t u_T(t,\cdot)\|_{\dot{H}^\ell(\mathbb{B}^5_{T-t})}} \lesssim (T-t)^\epsilon, \quad \ell \in \{0,1\}$$

Elements of proof

- Proof consists of perturbative construction around u_T
- Naive ansatz $u = u_T + \varphi$ leads to equation for φ with time-dependent coefficients
- Avoid this by introducing $\xi = \frac{x}{T-t}$ as a new spatial coordinate. With $\tau = -\log(T-t)$ as new time coordinate the resulting equation has τ-independent coefficients.
- With $v(\tau,\xi) = e^{-\tau}u(T-e^{-\tau}, e^{-\tau}\xi)$, the wave maps equation reads

$$\partial_\tau \begin{pmatrix} v(\tau,\cdot) \\ \partial_\tau v(\tau,\cdot) \end{pmatrix} = \mathbf{L}_0 \begin{pmatrix} v(\tau,\cdot) \\ \partial_\tau v(\tau,\cdot) \end{pmatrix} + \mathbf{F}\left(\begin{pmatrix} v(\tau,\cdot) \\ \partial_\tau v(\tau,\cdot) \end{pmatrix} \right)$$

  with a *spatial* differential operator $\mathbf{L}_0$
- Blowup solution u_T becomes

$$v_T(\tau,\xi) = e^{-\tau}u_T(T-e^{-\tau}, e^{-\tau}\xi) = \frac{2}{|\xi|}\arctan(|\xi|)$$

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$$\partial_\tau \Phi(\tau) = (\mathbf{L}_0 + \mathbf{L}')\Phi(\tau) + \mathbf{N}(\Phi(\tau))$$

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Analysis of the linearized operator

- $\mathbf{L}$ generates semigroup $\mathbf{S}(\tau)$ on $\mathcal{H} := H^2(\mathbb{B}^5) \times H^1(\mathbb{B}^5)$, i.e., $\Phi(\tau) = \mathbf{S}(\tau)\Phi(0)$ is solution of linearized equation

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- $\mathbf{L}$ is *nonself-adjoint*, spectral analysis requires sophisticated ODE tools and asymptotic resolvent estimates. Result:

$$\sigma(\mathbf{L}) = \{z \in \mathbb{C} : \operatorname{Re} z \leq -\epsilon\} \cup \{1\}$$

- Eigenvalue $1 \in \sigma(\mathbf{L})$ comes from freedom in choosing parameter T (blowup time). Define corresponding Riesz projection

$$\mathbf{P} = \frac{1}{2\pi i} \oint \mathbf{R}_{\mathbf{L}}(z) dz$$

and show $\operatorname{rank}(\mathbf{P}) = 1$ (spectral decomposition, stability of essential spectrum under compact perturbation, ODE analysis)

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End of proof

- Decomposition of linearized evolution by Gearhart-Prüss:

$$\mathbf{S}(\tau)\mathbf{P}\mathbf{f} = e^{\tau}\mathbf{P}\mathbf{f}$$
$$\|\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}} \lesssim e^{-\epsilon\tau}\|(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}}$$

- Nonlinearity $\mathbf{N}$ is locally Lipschitz, use Lyapunov-Perron method to construct co-dimension 1 manifold of data that lead to decaying evolution
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Two natural questions

- What happens outside the backward lightcone $\mathcal{C}_T$?
- What happens after blowup?

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Continuation beyond blowup

- Blowup solution can be smoothly extended beyond $t = T$:

$$\psi_T^*(t,x) = 4\arctan\left(\frac{r}{T-t+\sqrt{(T-t)^2+r^2}}\right)$$

solves

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right)\psi(t,r) + \frac{\sin(2\psi(t,r))}{r^2} = 0$$

- $\psi_T^*(t,r) = \psi_T(t,r)$ for $t < T$
- $\lim_{r\to 0+} \psi_T^*(t,r) = 2\pi$ if $t > T$ (change of topological charge due to blowup)
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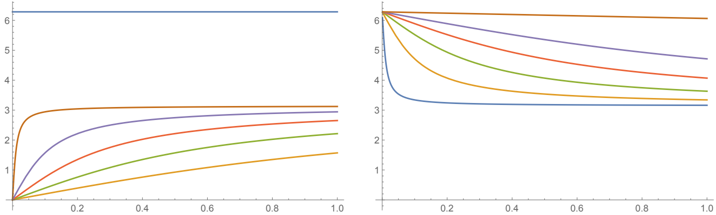
Snapshots of evolution



Hyperboloidal similarity coordinates

- Instead of similarity coordinates

$$\tau = T - e^{-\tau}, \quad x = e^{-\tau}\xi,$$

use *hyperboloidal similarity coordinates* (s, y) given by

$$t = T + e^{-s}h(y), \quad x = e^{-s}y, \quad h(y) = \sqrt{2 + |y|^2} - 2$$

- Still compatible with self-similarity: $\frac{x}{T-t} = -\frac{y}{h(y)}$
- Slices $s = \text{const}$ are curved and asymptotic to forward lightcones
- Coordinates (s, y) cover a large portion of spacetime almost up to the forward lightcone (Cauchy horizon) of the singularity
- Similar perturbative construction as in (τ, ξ) possible

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use *hyperboloidal similarity coordinates* (s, y) given by

$$t = T + e^{-s}h(y), \quad x = e^{-s}y, \quad h(y) = \sqrt{2 + |y|^2} - 2$$

- Still compatible with self-similarity: $\frac{x}{T-t} = -\frac{y}{h(y)}$
- Slices $s = \text{const}$ are curved and asymptotic to forward lightcones
- Coordinates (s, y) cover a large portion of spacetime almost up to the forward lightcone (Cauchy horizon) of the singularity
- Similar perturbative construction as in (τ, ξ) possible

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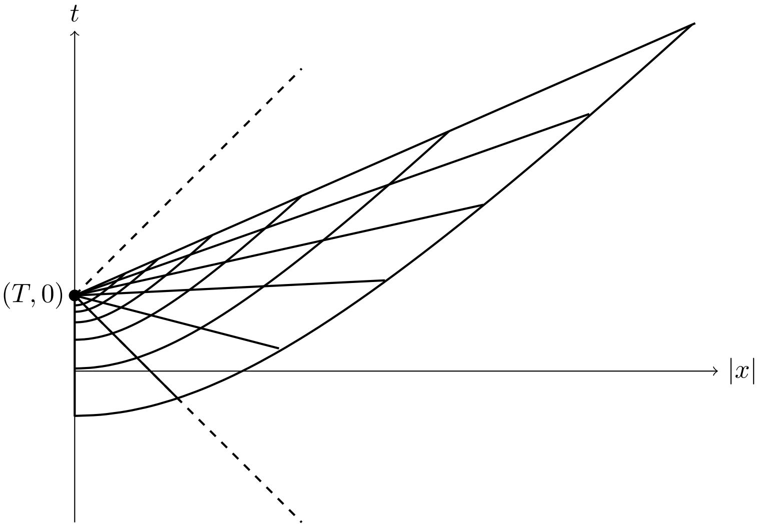
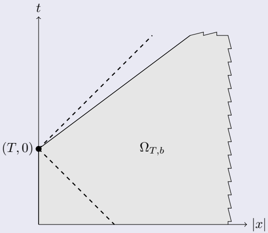
Geometry of HSC



Global stability of blowup

Theorem (Biernat, D., Schörkhuber 2017)

(f,g) smooth, radial, small in $H^m(\mathbb{R}^5) \times H^{m-1}(\mathbb{R}^5)$, $\operatorname{supp}(f,g) \subset \mathbb{B}^5_\epsilon \Rightarrow \exists\, T \approx 1$ and a unique smooth solution u with data $u(0,x) = u_1^*(0,x) + f(x)$, $\partial_0 u(0,x) = \partial_0 u_1^*(0,x) + g(x)$ in the domain $\Omega_{T,b}$



Global stability of blowup

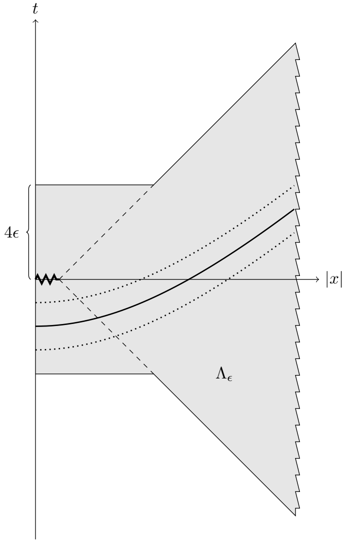
Theorem (continued)

Solution u *converges to* u_T^* *in the sense that*

$$e^{-s}\|(u\circ\eta_T)(s,\cdot)-(u_T^*\circ\eta_T)(s,\cdot)\|_{H^{m-3}(\mathbb{B}^5_R)}\lesssim e^{-\omega_0 s}$$
$$e^{-s}\|\partial_s(u\circ\eta_T)(s,\cdot)-\partial_s(u_T^*\circ\eta_T)(s,\cdot)\|_{H^{m-4}(\mathbb{B}^5_R)}\lesssim e^{-\omega_0 s},$$

where $u_T^*(t,x)=\frac{\psi_T^*(t,|x|)}{|x|}$ *and* $\eta_T(s,y)=(T+e^{-s}h(y),e^{-s}y)$

Idea of proof



Last slide

Thank you very much for your attention!