An efficient filtered scheme for some first order Hamilton-Jacobi-Bellman equations

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Outline

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2. Filtered scheme
3. Numerical Tests
4. Conclusion

Introduction

Time dependent HJ equation

We are interested in computing the approximation of viscosity solution of Hamilton-Jacobi (HJ) equation:

$$\begin{cases} \partial_t v + H(x, \nabla v) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (1)$$

$(A1)$ $H(x,p)$ is continuous in all its variables.
$(A2)$ $v_0(x)$ is Lipschitz continuous.

We aim to propose new higher order schemes, and prove their properties of consistency, stability and convergence.

Several schemes have been developed:

- Finite difference schemes (Crandall-Lions(84), Sethian(88), Osher/Shu(91), Tadmor/Lin(00)).
- Semi-Lagrangian schemes (Capuzzo Dolcetta(83,89,90)), (Falcone(94,09)/ Ferretti-Carlini(03,04,13)).
- Discontinuous Galerkin approach (Hu/Shu(99), Li/Shu(05), Bokanowski/Cheng/Shu(11,13,14), Cockburn(00).
- Finite Volume schemes (Kossioris/Makridakis/Souganidis(99), Kurganov/Tadmor(00)), Abgrall(00,01).

Monotone scheme

• **Discretization:** Let $\Delta t > 0$ denotes the time steps and $\Delta x > 0$ a mesh step, $t_n = n\Delta t$, $n \in [0, \dots, N]$, $N \in \mathbb{N}$ and $x_j = j\Delta x$, $j \in \mathbb{Z}$. For the given function $u(x)$.

Finite difference scheme (FD) Crandall-Lions (84) :

Let S^M be a **monotone FD scheme**

$$u^{n+1}(x_j) \equiv S^M(u_j^n) := u_j^n - \Delta t\, h^M(x_j, D^- u_j^n, D^+ u_j^n) \tag{2}$$

$$\text{with} \quad D^\pm u_j^n := \pm \frac{u_{j\pm 1}^n - u_j^n}{\Delta x}.$$

Assumptions on h^M :

$(A3)$ h^M is Lipschitz continuos function.

$(A4)$ (Consistency) $\forall x, \forall u$, $h^M(x, v, v) = H(x, v)$.

$(A5)$ (Monotonicity) for any functions u, v,

$$u \le v \Longrightarrow S^M(u) \le S^M(v).$$

Consistency error estimate: For any $v \in C^2([0,T] \times \mathbb{R})$, there exists a constant $C_M \geq 0$ independent of Δx such that

$$|\mathcal{E}_{S^M}(v)(t,x)| \leq C_M \Big(\Delta t \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \Big). \tag{3}$$

High Order scheme

Let S^A denote a **high order** (possibly unstable) scheme:

$$S^A(u^n)(x) := u^n(x) - \Delta t h^A(x, D^{k,-}u, \ldots, D^- u^n(x), D^+ u^n(x), \ldots, D^{k,+} u^n(x)) \tag{4}$$

High order consistency: There exists $k \geq 2$,and $1 < \ell < k$, for any $v = v(t,x)$ of class $C^{\ell+1}$, there exists $C_{A,\ell} \geq 0$,

$$|\mathcal{E}_{S^A}(v)(t,x)| \leq \ C_{A,\ell} \Big(\Delta t^\ell \|\partial_t^{\ell+1} v\|_\infty + \Delta x^\ell \|\partial_x^{\ell+1} v\|_\infty \Big). \tag{5}$$

Filtered scheme

It is known (Godunov's Theorem) that a monotone scheme can be at most of first order. Therefore it is needed to look for non-monotone schemes.

The difficulty is then to combine non-monotony with a convergence to viscosity solution of (1), and also obtain error estimates.

This is the core of the present work . In our approach we adapt an idea of Froese and Oberman (13) (for second order HJ equations) to treat mainly the case of evolutive first order PDEs.

Filtered scheme

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filtered scheme:

The scheme we propose is then

$$\boxed{u_j^{n+1} \equiv S^F(u_i^n) := S^M(u_j^n) + \epsilon \Delta t F\left(\frac{S^A(u_j^n) - S^M(u_j^n)}{\epsilon \Delta t}\right)} \quad (6)$$

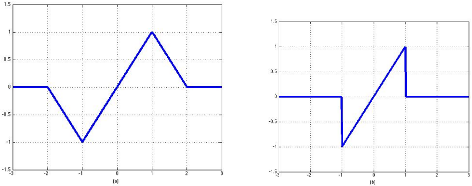
with a proper initialization of u_i^0.
• Where $\epsilon = \epsilon(\Delta t, \Delta x) > 0$ is the switching parameter that will satisfy

$$\lim_{(\Delta t, \Delta x) \to 0} \epsilon = 0.$$

More precision on the choice of ϵ will be given later on.

Filtered function

Instead of the Froese and Oberman's filter function i.e. $F(x) = sign(x)\max(1 - \big||x| - 1\big|, 0)$, we used new filter function i.e. $F(x) := x\,1_{|x|\leq 1}$:



Froese and Oberman's filter function and new filter function.

- To keep high order when $|h^A - h^M| \leq \epsilon$ i.e. $\left|\frac{S^A(u_j^n) - S^M(u_j^n)}{\epsilon\Delta t}\right| \leq 1 \Rightarrow S^F \equiv S^A$
- Otherwise $F = 0$ and $S^F = S^M$, i.e., the monotone scheme itself.

Consistency error estimate

For any regular function $v = v(t,x)$, for all $x \in \mathbb{R}$ and $t \in [0,T]$, we have

$$
\begin{aligned}
|\mathcal{E}_{S^F}(v)(t,x)| &= \left| \frac{v(t+\Delta t, x) - S^F(v(t,.))(x)}{\Delta t} - (v_t + H(x, v_x)) \right| \\
&\leq C_M \Big(\Delta t \|\partial_{tt} v\|_\infty + \Delta x \|\partial_{xx} v\|_\infty \Big) + \epsilon C_0 \Delta t.
\end{aligned} \tag{7}
$$

Definition (ϵ-monotonicity)

Filtered scheme is ϵ-monotone i.e. For any functions u, v,

$$u \leq v \Longrightarrow S(u) \leq S(v) + C\epsilon\Delta t,$$

where C is constant and $\epsilon \to 0$ as $\Delta t \to 0$.

Convergence Theorem

Theorem

Assume (A1)-(A2), and v_0 bounded. We assume also that S^M satisfies (A3)-(A5), and $|F| \leq 1$. Let u^n denote the filtered scheme (6)*. Let $v_j^n := v(t_n, x_j)$ where v is the exact solution of* (1)*. Assume*

$$0 < \epsilon \leq c_0 \sqrt{\Delta x} \tag{8}$$

for some constant $c_0 > 0$.
(i) The scheme u^n satisfies the Crandall-Lions estimate

$$\|u^n - v^n\|_\infty \leq C\sqrt{\Delta x}, \quad \forall\, n = 0, ..., N. \tag{9}$$

for some constant C independent of Δx.

Theorem ((Cont.))

(ii) *(First order convergence for classical solutions.) If furthermore the exact solution v belongs to $C^2([0, T] \times \mathbb{R})$, and $\epsilon \leq c_0 \Delta x$ (instead of* (8)*). Then it holds*

$$\|u^n - v^n\|_\infty \leq C\Delta x, \quad n = 0, ..., N, \tag{10}$$

for some constant C independent of Δx.

Theorem ((Cont.))

(iii) (Local high-order consistency.) Let $\mathcal{N}$ be a neighborhood of a point $(t, x) \in (0, T) \times \mathbb{R}$. Assume that S^A is a high order scheme satisfying (A7) for some $k \geq 2$. Let $1 \leq \ell \leq k$ and v be a $C^{\ell+1}$ function on $\mathcal{N}$. Assume that

$$(C_{A,1} + C_M) \left(\|v_{tt}\|_\infty \Delta t + \|v_{xx}\|_\infty \Delta x \right) \leq \epsilon. \quad (11)$$

Then, for sufficiently small $t_n - t$, $x_j - x$, Δt, Δx, it holds

$$S^F(v^n)_j = S^A(v^n)_j$$

and, in particular, a local high-order consistency error for the filtered scheme S^F:

$$\mathcal{E}_{S^F}(v^n)_j \equiv \mathcal{E}_{S^A}(v^n)_j = O(\Delta x^\ell)$$

(the consistency error $\mathcal{E}_{S^A}$ is defined as before).

Tuning of the parameter ϵ

Bounds for ϵ :

- **Upper bound:** $\epsilon \leq C\sqrt{\Delta x}$, with constant $C > 0$ ($\Rightarrow$ to have error estimates and convergence).
- **Lower bound:** $\frac{1}{2}|v_{xx}(x_i)| \left| \frac{\partial h^M(v)_i}{\partial Dv^+} - \frac{\partial h^M(v)_i}{\partial Dv^-} \right| \Delta x \leq \epsilon$. ($\Rightarrow$ to have high-order behavior)

$\Rightarrow$ Typically the choice $\epsilon := C\Delta x$ is made, for some constant C of the order of $\|v_{xx}\|_{L^\infty}$.

Eikonal equation in 1D

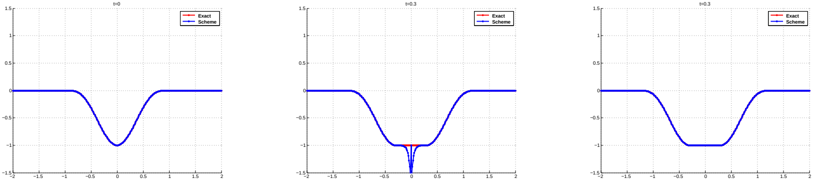
Example 2.

$$\begin{cases} v_t + |v_x| = 0, & t > 0,\ x \in (-2,2), \\ v(0,x) = v_0(x), := -\max(0, 1 - x^2)^4, & x \in (-2,2), \end{cases}$$

with periodic boundary condition on $(-2,2)$**, terminal time** $T = 0.3$

Errors		Filter $\epsilon = 5\Delta x$		CFD		ENO2	
M	N	L^2 error	order	L^2error	order	L^2 error	order
40	8	1.24E-02	1.93	2.02E-02	1.42	2.62E-02	1.49
80	16	3.05E-03	2.02	8.76E-03	1.20	8.08E-03	1.70
160	32	7.65E-04	2.00	1.04E-02	-0.24	2.52E-03	1.68
320	64	1.90E-04	2.01	1.23E-01	-3.57	7.88E-04	1.69
640	128	4.76E-05	2.00	1.03E+02	-9.70	2.47E-04	1.67

Table : L^2 errors for filtered scheme, Central finite difference (CFD) scheme, ENO (2nd order) scheme with RK2 in time.



Initial data (left), and plots at time $T = 0.3$, by Central finite difference scheme - middle - and Filtered scheme - right ($M = 160$ mesh points).

1D steady equation

Example 3. (as in Abgrall [6]) $\begin{cases} |v_x| = f(x), \text{ on } (0,1) \\ v(0) = v(1) = 0 \end{cases}$ $f(x) := 3x^2 + a$, $a := \frac{1-2x_0^3}{2x_0 - 1}$, $x_0 := \frac{\sqrt[3]{2}+2}{4\sqrt[3]{2}}$.

	Filter ($\epsilon = 5\Delta x$)		CFD		ENO	
M	L^∞ error	order	L^∞ error	order	L^∞ error	order
100	1.15E-03	2.25	$+$inf	-	3.54E-02	0.99
200	2.68E-04	2.10	$+$inf	-	1.78E-02	1.00
400	8.74E-05	1.62	$+$inf	-	8.89E-03	1.00
800	4.20E-05	1.06	$+$inf	-	4.45E-03	1.00

Table : L^∞ Errors for filtered scheme, CFD scheme, and RK2-2nd order ENO scheme. $\Rightarrow$ **Filter can stabilize an otherwise unstable scheme**

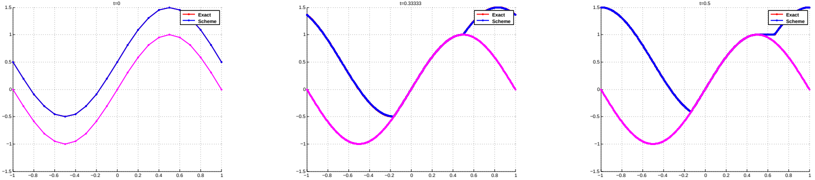
Advection with obstacle

Example 4. (as in Bokanowski et al. [9]**)** $\begin{cases} \min(v_t + v_x, g(x)) = 0, & t > 0, x \in [-1, 1], \\ u(0, x) = 0.5 + sin(\pi x) & x \in [-1, 1], \end{cases}$

with periodic boundary condition, $g(x) = sin(\pi x)$**, and where the terminal time** $T = 0.5$**.**

Errors		Filter $\epsilon = 5\Delta x$		CFD		ENO2	
M	N	L^∞ error	order	L^∞ error	order	L^∞ error	order
40	20	7.93E-03	2.03	1.63E-02	1.54	2.14E-02	1.59
80	40	1.84E-03	2.10	2.98E-02	-0.87	7.75E-03	1.46
160	80	3.92E-04	2.24	4.46E-02	1.03	1.07E-03	2.86
320	160	9.67E-05	2.02	8.02E-03	0.86	2.72E-04	1.97
640	320	2.40E-05	2.01	4.10E-03	0.97	6.92E-05	1.98

Table : Local L^∞ errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.



Initial data (left), and plots at time $T = 0.3$ - middle and $T = 0.5$- right by filtered scheme.

Eikonal in 2D

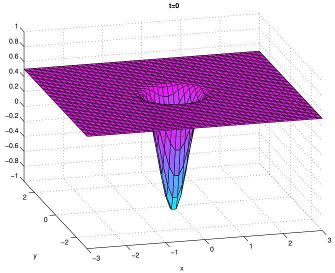
Example 6. In this example we consider HJB equation with smooth initial data and $\Omega = (-3,3)^2$

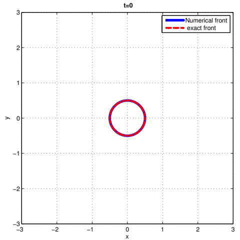
$$\begin{cases} v_t + |\nabla v| = 0, \quad (x,y) \in \Omega,\ t > 0, \\ v(0,x,y) = v_0(x,y) = v_0(x,y) := 0.5 - 0.5 \max(0, \frac{1-(x-1)^2-y^2}{1-r_0^2})^4, \end{cases}$$

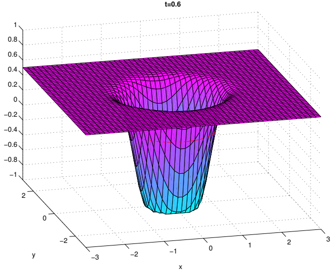
where $|.|$ **is the Euclidean norm and** $r_0 = 0.5$ **with Drichlet boundary conditions.**

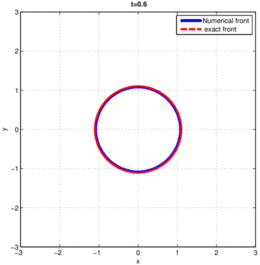
		Filter ($\epsilon = 20\Delta x$)		CFD		ENO2	
$Mx = Nx$	N	L^2 error	order	L^2 error	order	L^2 error	order
25	25	3.39E-01	-	4.54E-01	-	4.22E-01	-
50	50	1.14E-01	1.57	2.11E-01	1.11	1.57E-01	1.42
100	100	2.77E-02	2.04	8.89E-02	1.24	5.12E-02	1.62
200	200	6.81E-03	2.02	3.99E-02	1.16	.48E-02	1.80
400	400	1.70E-03	2.00	1.87E-02	1.10	4.34E-03	1.77

Table : L^2 errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.


t=0
x
y


t=0
Numerical front
exact front
x
y


t=0.6
x
y


t=0.6
Numerical front
exact front
x
y

Eikonal in 2D

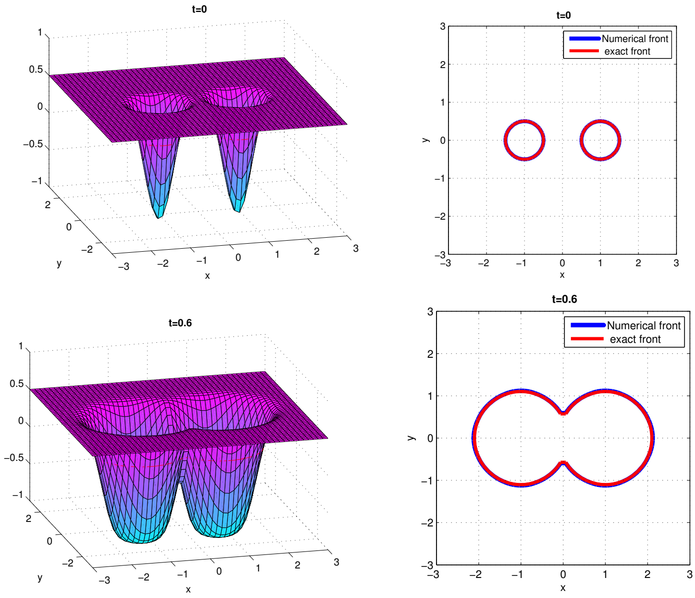
Example 7. $\Omega = (-3,3)^2$

$$\left\{ v(0,x,y) = v_0(x,y) = 0.5 - 0.5\max\left(\max(0, \frac{1-(x-1)^2-y^2}{1-r_0^2})^4, \max(0, \frac{1-(x+1)^2-y^2}{1-r_0^2})^4 \right). \right.$$

where $|.|$ **is the Euclidean norm and** $A_\pm := (\pm 1, 0)$ **with Drichlet boundary conditions and CFL condition** $\mu = 0.37$

		Filter ($\epsilon = 20\Delta x$)		CFD		ENO2	
$Mx = Nx$	N	L^2 error	order	L^2 error	order	L^2 error	order
25	25	5.39E-01	-	3.73E-01	-	4.22E-01	-
50	50	1.82E-01	1.57	1.42E-01	1.39	1.57E-01	1.42
100	100	3.72E-02	2.29	4.72E-02	1.59	5.12E-02	1.62
200	200	9.36E-03	1.99	1.66E-02	1.51	1.48E-02	1.80
400	400	2.36E-03	1.99	7.23E-03	1.20	4.34E-03	1.77

Table : L^2 errors for filtered scheme, Central finite difference (CFD) scheme, ENO (second order) scheme and RK2 in time.


t=0
t=0
Numerical front
exact front
t=0.6
t=0.6
Numerical front
exact front
x
y

Conclusion

• A general and simple presentation of filtered scheme and easy to implement.
• Convergence of filtered scheme is confirmed. Error estimate is of $O(\sqrt{\Delta x})$ and numerically observed that $O(\Delta x^2)$ behavior in smooth regions.
• Remark : We propose a general strategy of taking a good scheme (like ENO second order) but for which there is no convergence proof, and use the filter to assure convergence and error estimate (the theoretical $\sqrt{\Delta x}$ as for the monotone scheme), and numerically show that we almost keep the same precision as ENO (i.e. second order) on basic linear and non linear examples.

Tanti Auguri



Organizing team.