Costly Global Asymptotic Controllability (Si ad metam gratuitus non est accesus)

Franco Rampazzo

Università di Padova, Italy

Maurizio Falcone's 60th birthday
December 4-5, 2014
Università di Roma "La Sapienza"

A control system

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- Write $x[z, u]$ for the solution(s) corresponding to initial state z and control u.

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that is

Non-Costly (=**Free of charge**)
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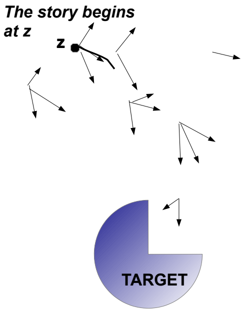
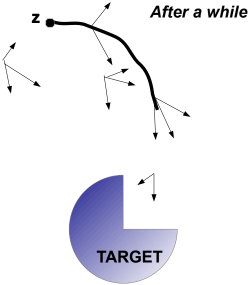


Figure : Global Asymptotic Controllability



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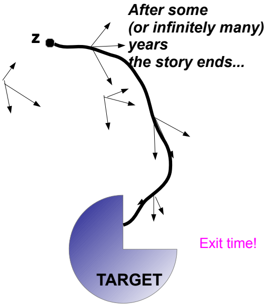

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Definition. *The system is* **globally asymptotically controllable** (GAC) *provided there is a function* $\beta \in \mathcal{KL}$ *such that, for each initial state* $z \in \Omega \setminus \mathcal{T}$, *there exists an admissible trajectory-control pair* $(x, u) : [0, +\infty[\to \mathbb{R}^n \times U$ *that verifies*

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where $\beta \in \mathcal{KL}$ means:
(1) $\beta(0, t) = 0$ and $\beta(\cdot, t)$ is strictly increasing;
(2) $\beta(r, \cdot)$ is decreasing and $\beta(r, t) \to 0$ as $t \to +\infty$.

DEFINITION. $V : \mathbb{R}^n \setminus \overset{\circ}{\mathbf{C}} \to \mathbb{R}$ is a **Control Liapunov Function** (**CLF**), if

- V is continuous, locally semiconcave, positive definite, proper on $\mathbb{R}^n \setminus \mathbf{C}$;
- and

$$H^{\mathcal{F}}(z,p) < 0 \qquad \forall p \in D^* V(z)$$

where $H^{\mathcal{F}}$ is the Hamiltonian associated with the vector field $\mathcal{F}$, namely,

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$D^*V(z)$ denotes the *set of limiting gradients* of V at z:
$D^*V(z) \doteq \left\{ w : \ w = \lim_k \nabla V(z_k),\ z_k \in DIFF(V) \setminus \{z\},\ \lim_k z_k = z \right\}.$

THEOREM:
IF THERE EXISTS A CONTROL LIAPUNOV FUNCTION V THEN THE SYSTEM IS GAC.

See works on feedback stabilization and input-to-state stability [Clarke, Ledyaev, Sontag, Subbotin, 97], [Malisoff, Rifford, Sontag, 04].

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Remark: Some converse statements are true as well, but this requires much care, for the whole Lie brackets stuff should matter at some point...

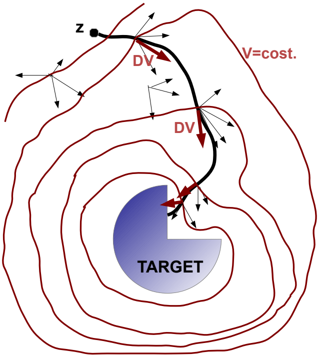

Figure : Level sets of a Control Liapunov Function.

LIAPUNOV

Liapunov

Liapunov

Liapunovs0.pdf

The LIAPUNOVS

Liapunov,**Serjei**, Russian MUSICIAN

Liapunov,**Aleksandr,** MATHEMATICIAN,
Serjei's brother

Liapunovs.pdf

A third Lyapunov (Aleksandr's nephew?)

third.pdf

Figure : Aleksey Lyapunov (range of vector measures)

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Notice that $\mathcal{T}(z) = +\infty$ means that

- either one cannot even "approach" the target from z
- or the target cam be "approached" asymptotically from z at $t = +\infty$.

Some known results

Minimum time is the most studied exit-time optimal control problem (see e.g., [Cannarsa, Sinestrari, 04]). Use **d** to denote the *distance* from **C** and let D^* be the it limiting gradient.

THEOREM. Assume **Petrov condition**
(P) $\exists \delta, \mu > 0$ such that such that

$$\boxed{\min_{u \in U} H^{\mathcal{F}}(z, p) < -\mu} \qquad \forall p \in D^* \mathbf{d}$$

for all $z \in B(\mathbf{C}, \delta)$. **Then**: the minimum time function $T(z)$ is Lipschitz continuous; in particular,

$$T(z) \leq K\, \mathbf{d}(z)$$

near the target.

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$$\boxed{\min_{u\in U}\langle \mathcal{F}(z, a), D^*\mathbf{d}(z)\rangle \leq -\mu(\mathbf{d}(z))}$$

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Then:

$T(z)$ is continuious near **C**; moreover, there exists $K > 0$ such that

$$T(z) \leq K \, \Phi(\mathbf{d}(z)),$$

where $\Phi(r) \doteq \int_0^r \frac{d\rho}{\mu(\rho)}$, $r > 0$.

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(when $\mu(r) = r^{\frac{1}{3}}$, then $\Phi(r) = 3/2 r^{\frac{2}{3}}$)

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Theorem. If T is continuous on $\partial\mathbf{C}$, then T is the unique continuous viscosity solution of

$$-H(z, Du) \doteq -\min_{u\in U}\{\langle Du, \mathcal{F}(z,a)\rangle + 1\} = 0 \quad \text{in } \mathcal{R}\setminus \mathbf{C}$$

such that $u = 0$ on $\partial\mathbf{C}$ and $\lim_{z\to\bar{z}} u(z) = +\infty \quad \forall \bar{z} \in \partial\mathcal{R}$.

($\mathcal{R}$ denotes the *reachable set*)

Remark. Setting $V(z) \doteq \mathbf{d}(z)/\mu$ and $V(z) \doteq \Phi(\mathbf{d}(z))$, respectively, conditions **(P)**(Petrov) and **(WP)**(weak Petrov) can rephrased as follows:

if $\mathbf{d}(z) < \delta$,stated

$$\min_{u \in U} \Big\langle (p_{\mathcal{I}}, p)\, ,\, (1, \mathcal{F}(z,a)) \Big\rangle \leq 0 \qquad \forall p \in D^* V(z)$$

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- Recipes to construct feedback

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(The heuristics is: "*use a clock with state-control dependent speed* $\frac{1}{l(x,a)}$ *instead of a uniform universal clock with speed* 1")

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The "degenerate clock" problem

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- Furthermore, $\mathcal{F}$, l continuous on $(\mathbb{R}^n \setminus \mathbf{C}) \times U$

Various consequences of degeneracy ($l \geq 0$)

- lack of uniqueness for the associated PDE:

$$-\max_{u\in U}\{\langle Du(z,a),\mathcal{F}(z,a)\rangle + l(z,a)\} = 0$$

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- "Lavrentiev" phenomenon for unbounded and impulsive controls:
  [Guerra, Sarychev, '09], [Motta, Sartori,'11],[Aronna,Motta,Rampazzo),'14]

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- **Global Asymptotic Controllability (GAC)**
- **Bounds and regularity on the boundary for the Value Function**

Specializing Control Liapunov Functions: Minimum Restraint Functions

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NOTICE THAT THE INEQUALITY IS STRICT

Remark 1. MRF includes Petrov and Weak Petrov introduced before.
For instance, Petrov condition reads,

$$\min_{u \in U} \{\langle p, \mathcal{F}(z,a)\rangle + \mu\} \leq 0 \quad \forall p \in D^* V(z),$$

which, setting $l(x,u) := 1/2\mu$, can be rephrased as

$$\min_{u \in U} \Big\langle (p_{\mathcal{I}}, p), (l(x,u), \mathcal{F}(z,u)) \Big\rangle \leq -(1/2)\mu < 0 \quad \forall p \in D^* V(z)$$

with $p_{\mathcal{I}} = 1$.

- **Remark 2.** If V is a Minimum Restraint Function, then V is a Control Lyapunov Function.

- Indeed, from

$$\min_{u\in U}\{p_{\mathcal{I}}\, l(z,a) + \langle D^* V(z)\,,\, \mathcal{F}(z,a)\rangle\} < 0 \text{ and } l(z,a) \geq 0$$

we get

$$\min_{u\in U}\langle D^* V(z), \mathcal{F}(z,a)\rangle < 0$$

Theorem.[Motta-Rampazzo '13]
Assume the control set U is bounded.
Let V be a MRF. Then

i) *the system is Globally Asymptotically Controllable*
ii) furthermore, if the savings multiplier $p_{\mathcal{I}}$ is > 0, the Value Function $\mathcal{W}$ verifies

$$\mathcal{W}(z) \leq \frac{V(z)}{p_{\mathcal{I}}}.$$

Ingredients of the proof

- The proof of Theorem 1 is based on the following

Proposition 1. Let V be a MRF. Then $\forall \sigma > 0$ there exists a continuous, strictly increasing function $m : [0, +\infty[\to \mathbb{R}$, verifying $m(r) > 0 \quad \forall r > 0$, such that, setting

$$g(z, a) \doteq k\, l(z, a) + m(V(z)),$$

for all $(z, a) \in \, V^{-1}(]0, \sigma]) \times U$ one has

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Ingredients of the proof

- The proof of Theorem 1 is based on the following

Proposition 1. Let V be a MRF. Then $\forall\sigma > 0$ there exists a continuous, strictly increasing function $m : [0, +\infty[\to \mathbb{R}$, verifying $m(r) > 0 \quad \forall r > 0$, such that, setting

$$g(z,a) \doteq k\, l(z,a) + m(V(z)),$$

for all $(z,a) \in \ V^{-1}(]0,\sigma]) \times U$ one has

$$\min_{u\in U}\left\{\langle D^* V(z), \mathcal{F}(z,a)\rangle + g(z,a)\right\} \leq 0.$$

- Notice that $g(z,a) \geq k\, l(z,a)$ and $\quad g(z,a) > 0$ outside the target.

Thanks to Proposition 1,
the upper bound in Theorem 1 can be improved:

$$\mathcal{W}(z) \leq \int_0^{t_z(x)} l(x(t), u(t))\, dt \leq \frac{1}{k} \int_0^{t_z(x)} g(x(t), u(t))\, dt \leq \frac{V(z)}{k}.$$

Ingredients of the proof

- The proof of Theorem 1 relies also on
  - the construction of a discontinuous feedback control law;
  - the use of the semiconcavity property of the MRF V,

  in the spirit of feedback stabilization and input-to-state stability
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- The question of **feedback stabilization and input-to-state stability <u>with a cost</u>** is a natural future issue.

Uniqueness

From Theorem 1 one can derive **explicit sufficient conditions in order to characterize** $\mathcal{W}$ as unique, nonnegative continuous viscosity solution of

$$-\min_{u \in U} \{\langle \mathcal{F}(z,a), Du \rangle + l(z,a)\} = 0 \quad \text{in} \quad Dom(\mathcal{W}) \setminus \mathbf{C}$$

such that $u = 0$ on $\partial\mathbf{C}$ and $\lim_{z \to \bar{z}} u(z) = +\infty \quad \forall \bar{z} \in \partial Dom(\mathcal{W})$.

($Dom(\mathcal{W}) \doteq \{z : \ \mathcal{W}(z) < +\infty\}$ denotes the domain of $\mathcal{W}$)

[Motta, '04], [Motta, Sartori, in preparation]

Approximations

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or

- both conditions together.

What happens with UNBOUNDED CONTROLS?

Joint, current, work with **ANNA CHIARA LAI**

Let us point out that

- Compactness of U was essential in the proof of the main theorem, in particular in the implementation of the *hold-and-sample method* to prove GAC.
- In many applications (but also in Calculus of Variations!) the L^∞ boundedness of controls IS NOT a natural hypothesis
- In partucular the dynamics $\mathcal{F}$ can be POLYNOMIAL IN u

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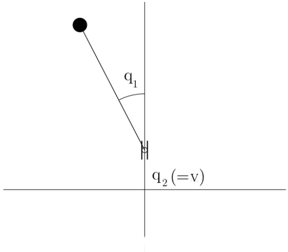
- In many applications (but also in Calculus of Variations!) the L^∞ boundedness of controls IS NOT a natural hypothesis
- In partucular the dynamics $\mathcal{F}$ can be POLYNOMIAL IN u (take advantage of algebraic structure?)

Motivations: an example from mechanics

Inverted pendulum with oscillating pivot.



In presence of the gravity force g, the control equations for q_1 and for the corresponding momentum p_1 are

$$\begin{cases} \dot{q}_1 = p + \sin(q_1)\dot{v} \\ \dot{p}_1 = g\sin(q_1) - p_1\cos(q_1)\dot{v} - \sin(q_1)\cos(q_1)\dot{v}^2 \end{cases}$$

Setting $x = (p_1, q_1, v)$ and $u = \dot{v}$, we obtain a **control quadratic system** of the form

$$\dot{x} = f(x) + g(x)u + h(x)u^2$$

More general mechanical motivations: mechanical system controlled by moving constraints are unbounded control-quadratic systems [Bressan, Rampazzo,Arch.Rat.Mech.and Anal]

Hamiltonian equations of motion

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = f(p,q) + \sum_{\alpha=1}^{m} g_\alpha(p,q)\dot{v}_\alpha + \sum_{\alpha,\beta=1}^{m} g_{\alpha,\beta}(p,q)\dot{v}_\alpha \dot{v}_\beta$$

with suitable vector fields $f, g_\alpha, g_{\alpha\beta}$ determined by the Kinetic Energy and by the applied forces.

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with suitable vector fields $f, g_\alpha, g_{\alpha\beta}$ determined by the Kinetic Energy and by the applied forces.
Setting $x = (p_1, q_1, q_2)$ and $u = \dot v$ we obtain **the control-quadratic system**

$$\dot x = f(x) + \sum_{\alpha=1}^{m} g_\alpha(x)u_\alpha + \sum_{\alpha,\beta=1}^{m} g_{\alpha,\beta}(x)u_\alpha u_\beta$$

Main assumption in the unbounded control case

Hypothesis $\mathbf{A}_{main}$: *For every compact subset $K \subset \mathbb{R}^n$ the function*

$$(\bar{l}, \bar{\mathcal{F}})(x, u) := \frac{(l, \mathcal{F})(x, u)}{1 + |(l, \mathcal{F})(x, u)|}$$

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Hypothesis $\mathbf{A}_{main}$ is *quite weak*: it allows for e.g. (x-dependent) polynomials in $u_1, \cdots, u_m$, $|u_1|, \cdots, |u_m|$, $|u|$ or compositions of polynomials with exponential and Lipschitz continuous functions.

Theorem (Lai-Rampazzo)

Let V be a Minimum Restraint Function and assume Hypothesis $\mathbf{A}_{main}$. *Then:*

(i) *the system $\mathcal{F}$ is globally asymptotically controllable to $\mathcal{T}$;*

(ii) *if V has savings multiplier $\bar{p}_{\mathcal{I}} > 0$, then*

$$W(x) \leq \frac{V(x)}{\bar{p}_{\mathcal{I}}}$$

Dynamics which are *polynomial in the control* $u \in \mathbb{R}^m$:

$$\mathcal{F}(x,u) := f(x) + \sum_{\alpha=1}^{m} u_\alpha g_\alpha(x) + \cdots + \sum_{\alpha_1 \le \cdots \le \alpha_d} u_{\alpha_1} \cdots u_{\alpha_d} g_{\alpha_1 \dots \alpha_d}(x).$$

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Investigate algebraic properties of the **convex hull**

$$co\ \mathcal{F}(x, \mathbb{R}^m)$$

In particular:

(Q1) $co\ \mathcal{F}(x, \mathbb{R}^m) = \mathbb{R}^M$?

(Q2) Can we find "simple" selections of $co\ \mathcal{F}(x, \mathbb{R}^m)$?

1- Control-polynomial systems which can be "represented" by affine-control systems

Because of nonlinearity, this is false in genera: $\dot{x} = f(x) + h(x)u^2$, $u \in \mathbb{R}$. On the other hand, a system of the form

$$\dot{x} = f(x) + g_1(x)u_1 + g_{1,3}u_1u_3^5 + g_{2,6}u_2^3u_6^3 + g_{1,3,7}u_1u_3^5u_7^9 \qquad u \in \mathbb{R}^7$$

can be represented as

$$\dot{x} = f(x) + g_1(x)w_1 + g_{1,3}(x)w_2 + g_{2,6}w_3 + g_{1,3,7}(x)w_4 \qquad w \in \mathbb{R}^4$$

QUESTION 1: AFFINE REPRESENTABILITY

Can one represent

$$\mathcal{F}(x,u) := f(x)+\sum_{\alpha=1}^{m} u_\alpha g_\alpha(x)+\cdots+\sum_{\alpha_1\leq\cdots\leq\alpha_d} u_{\alpha_1}\cdots u_{\alpha_d} g_{\alpha_1\ldots\alpha_d}(x).$$

with the affine associated system

$$\mathcal{F}_{aff}(x,w) := f(x)+\sum_{\alpha_1} w_{\alpha_1} g_{\alpha_1}(x)+\cdots+\sum_{\alpha_1<\cdots<\alpha_d} w_{\alpha_1\cdots\alpha_d} g_{\alpha_1\ldots\alpha_d}(x) \; ?$$

YES, if the system is "balanced"

Definition (Balanced systems)

We say that the control-polynomial dynamics is *balanced* if there exist an $m-$tuple $K = (K_1, \ldots, K_m)$ of positive odd numbers and a positive integer number $\bar{d} \le d$ such that

$$\mathcal{F}(x,u) = f(x) + \sum_{\alpha_1} u^K_{\alpha_1} g_{\alpha_1}(x) + \cdots + \sum_{\alpha_1 < \cdots < \alpha_{\bar{d}}} u^K_{\alpha_1} \cdots u^K_{\alpha_{\bar{d}}} g_{\alpha_1 \ldots \alpha_{\bar{d}}}(x),$$

where we have set $u^K_\alpha := u^{K_\alpha}_\alpha$.

QUESTION 2: WEAK SUBSYSTEMS

Can we single out simple *Weak subsystems* to which we can apply the general theorem?

A *weak subsystems* is a parametrized selections of the set-valued function $x \mapsto co\ \mathcal{F}(x, \mathbb{R}^m)$

An answer

YES, for instance

THE MAXIMAL DEGREE SUBSYSTEM:

$$\mathcal{F}^{max}(x,u) := f(x) + \sum_{\alpha_1 \leq \cdots \leq \alpha_d} u_{\alpha_1} \cdots u_{\alpha_d} g_{\alpha_1 \dots \alpha_d}(x).$$

HAPPY BIRTHDAY, DEAR MAURIZIO!

Ulteriora mirari, praesentia sequi