Stabilization with discounted optimal control

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based on joint work with

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Setup

We consider continuous time finite dimensional control systems

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, $\quad u \in \mathcal{U} = L^\infty(\mathbb{R}, U)$

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Goal: Given an equilibrium x^e (i.e., $f(x^e, u^e) = 0$ for some $u^e \in U$), for any initial value x_0 find a control which steers the trajectory to x^e and keeps it there

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Approach: Compute this u via optimal control, preferably in feedback form $u(t) = F(x(t))$

Special case: linear quadratic optimal control

For the special case of linear systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$

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$$\underset{u\in\mathcal{U}}{\text{minimize}} \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

with matrices $R > 0$, $Q > 0$ yields such controls.

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This linear quadratic problem is efficiently solvable via the algebraic Riccati equation

Nonlinear case

In the nonlinear case

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0$$

under standard regularity assumptions, stabilization can be achieved via the optimal control problem

$$\underset{u \in \mathcal{U}}{\text{minimize}} \int_0^\infty \ell(x(t), u(t))dt$$

with ℓ satisfying $\ell(x,u) > 0$ whenever $x \neq x^e$

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Drawback: this problem is very difficult to solve

Solution strategies

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Receding Horizon (aka Model Predictive) Control:

For $i = 0, 1, 2, \ldots$ solve iteratively

$$\underset{u \in \mathcal{U}}{\text{minimize}} \int_{t_i}^{t_i+T} \ell(x(t), u(t))dt$$

and apply the optimal control on $[t_i, t_{i+1}]$ (usually $t_{i+1} \ll t_i + T$)

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Disadvantage: very hard to solve for large T (may be necessary to ensure stability)

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Transform the problem via the Kružkov transform

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Disadvantage: Hamilton-Jacobi-Bellman equation has a singularity at $x^e \rightsquigarrow$ control only stabilizes a neighborhood of x^e

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$$\text{minimize} \int_0^{t(x,u)} \ell(x(t), u(t))dt$$

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Disadvantage: again, the resulting control only stabilizes a neighborhood of x^e (including the target)

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New idea:

Solve a discounted problem with $\delta > 0$

$$\text{minimize} \int_0^\infty e^{-\delta t} \ell(x(t), u(t))dt$$

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Question: will the optimal control of the discounted problem stabilize the system?

Notation and assumptions

We define the discounted functional

$$J_\delta(x_0,u) := \int_0^\infty e^{-\delta t}\ell(x(t),u(t))dt$$

and the optimal value function

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For simplicity, we do not consider state constraints in this talk (but results can be extended provided V_δ remains continuous)

Towards a sufficient condition

For model predictive control, consider the undiscounted optimal value function

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$$V_0(x_0) = \inf_{u \in \mathcal{U}} \int_0^\infty \ell(x(t), u(t)) dt$$

Then, model predictive control stabilizes the equilibrium if the inequality

$$V_0(x_0) \leq \gamma \min_{u \in U} \ell(x_0, u)$$

holds for some $\gamma > 0$ and all $x_0 \in \mathbb{R}^n$

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The larger γ, the larger T must be for guaranteeing stability

Main theorem

Theorem: Assume that

(i) V_δ satisfies the inequality

$$\alpha_1(\|x - x^e\|) \le V_\delta(x) \le \alpha_2(\|x - x^e\|)$$

for functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$

(ii) there exists $K > \delta$ such that the inequality

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Then the discounted optimal control stabilizes the equilibrium x^e

Idea of proof

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For almost all $t \geq 0$, the dynamic programming principle and (ii) $KV_\delta(x) \leq \min_{u\in U} \ell(x,u)$ with $K > \delta$ imply:

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$$\Rightarrow \qquad V_\delta(x^\star(t)) \leq e^{-(K-\delta)t} V_\delta(x_0)$$

Together with the bounds (i) on V_δ, this implies the claimed asymptotic stability

Approximately optimal trajectories

The statement can be extended to approximately optimal controls $\tilde{u}^\star$

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Approximately optimal trajectories

The statement can be extended to approximately optimal controls $\tilde{u}^\star$, provided the relative error along the trajectory $\tilde{x}^\star$, i.e.,

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is sufficiently small for all t

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Discussion of the conditions

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These bounds can be assured by appropriate choice of ℓ:

ℓ must be sufficiently flat near x^e, sufficiently large away from x^e and fast dynamics must be penalized sufficiently strong

Discussion of condition (ii)

Assumption (ii): There exists $K > \delta$ with

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For $\delta < 1/\gamma$, this inequality follows from the stability condition for model predictive control

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For $\delta < 1/\gamma$, this inequality follows from the stability condition for model predictive control

$$V_0(x) \leq \gamma \min_{u \in U} \ell(x,u)$$

This condition, in turn, is always satisfied for suitable ℓ if the system is finite time or exponentially controllable to x^e

Example

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_2 - x_1 x_2\end{aligned}$$

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$\rightsquigarrow$ predator-prey model (x_1 = predator, x_2 = prey) which for $u = 0$ has an equilibrium at $(1, 1)^T$ and periodic trajectories.

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The u-term models that the predators are hunted for

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The goal is to stabilize $x^e = (1, 1.26)^T$ which is an equilibrium for $u^e = 0.26$.

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Numerical computations were performed for different δ using the occupational measure approach of V. Gaitsgory et al.

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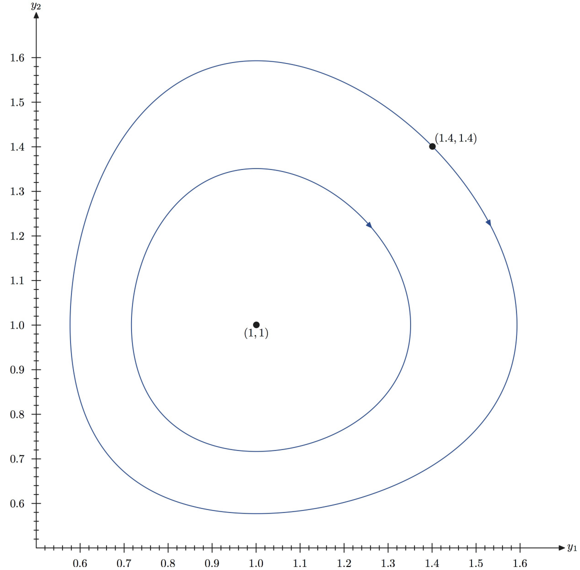
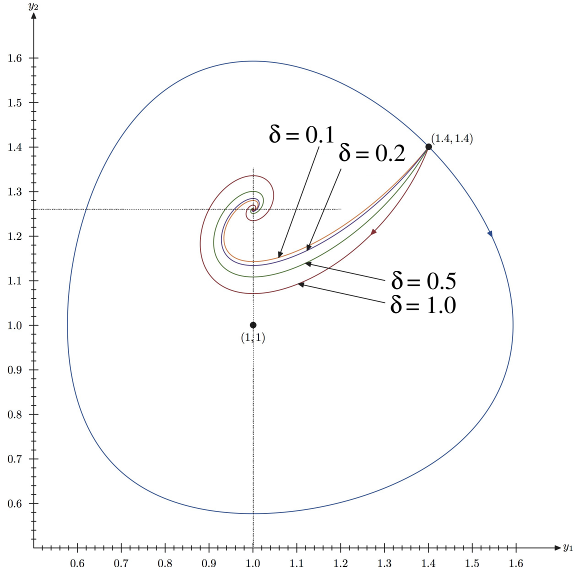


Uncontrolled

Example



Stabilized at $x^e = (1, 1.26)^T$

Conclusions

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- Reference: V. Gaitsgory, L. Grüne, N. Thatcher
  Stabilization with discounted optimal control
  Preprint available from `num.math.uni-bayreuth.de`

Auguri, Maurizio!