Eikonal equations on the Sierpinski gasket[1]

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[1] F. CAMILLI, R.CAPITANELLI, C. MARCHI, Eikonal equations on the Sierpinski gasket, arXiv:1404.3692, 2014

HJ equations and viscosity solutions in “non regular” settings:

- **Non regular geometric structures (networks, stratified sets, ramified spaces)**
  - Y.ACHDOU, F.CAMILLI, A.CUTRÌ,N.TCHOU, Hamilton-Jacobi equations constrained on networks, NoDEA (2013).
  - C.IMBERT, R. MONNEAU, H. ZIDANI, A Hamilton-Jacobi approach to junction problems and application to traffic flows, COCV (2013)
  - D. SCHIEBORN, F.CAMILLI, Viscosity solutions of eikonal equations on topological networks, CVPDE (2013).
  - G. BARLES, A. BRIANI, E. CHASSEIGNE, A Bellman approach for two-domains optimal control problems in $\mathbb{R}^N$, COCV (2014)
- **Spaces of probability measures and metric spaces**
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  - GIGA-HAMAMUKI-NAKAYASU, Eikonal equations in metric space, Trans.AMS, 2015.

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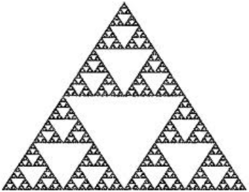
Aim of this talk is to present a notion of solution for the Eikonal equation

$$|Du| = f(x) \qquad x \in S$$

where S is the Sierpinski gasket (or a post critically finite fractal).

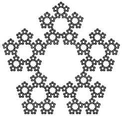


(a) Sierpinski gasket

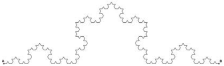
Fractal sets are an intermediate case since:

- Fractals can be seen as the limit case of nonregular geometric sets (prefractals)
- As for metric spaces, there is no natural notion of gradient on fractals.

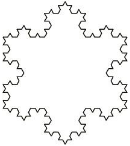
Fractals



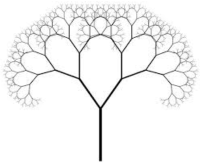
(b) Penta-gasket



(c) Koch curve



(d) Koch island



(e) Tree

The term fractal is derived from the Latin adjective *fractus*, which means broken in irregular fragments. The main features are

- Self-similarity - when broken into smaller and smaller pieces, the new pieces look exactly the same as the original
- Dimension - the fractals are curves that fill the space

A brief history

- **K. W. T. Weierstrass** (1815-1897): Nowhere Differentiable Functions
- **G.Cantor**: Cantor set
- **Sierpinski, Julia, Koch** (beginning of '900): fractal set via the iteration of a function of a complex variable
- **P. Levy** : Fractal Random Walks (Random Fractals)
- **B. Mandelbrot** (1970s:) Mandelbrot sets and the development of a general theory on the "Fractal Geometry of Nature" (1975)

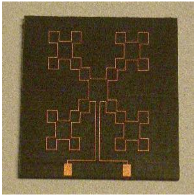
A dreadful plague

After Weierstrass invented his nowhere differentiable function, many mathematicians were alarmed at losing the property of differentiation as a constant.
Hermite described these new functions as a "dreadful plague" and **Poincaré** wrote

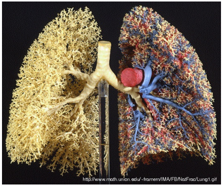
"Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show up the arguments of our fathers, and they will never have any other use".

Applications of fractals

- **Signal Processing:** Time Series Analysis, Speech Recognition
- **Image Processing:** Fractal Compression, Fractal Dimension Segmentation
- **Simulation:** Terrain Modeling, Image Synthesis, Music, Stochastic Fields
- **Financial:** Fractal Market Analysis, Futures Markets
- **Medicine:** Histology, Monitoring, Epidemiology
- **Military:** Visual Camouflage, Covert Digital Communications



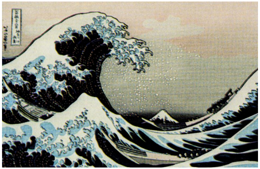
(f) Mobile antenna



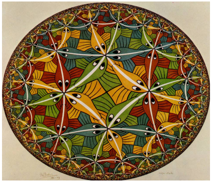
(g) lung

(h) camouflage

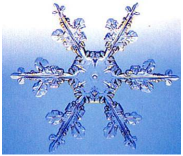
Fractals everywhere



(i) Hokusai



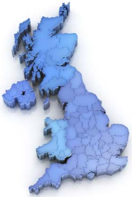
(j) Escher



(k) Snow flake

(l) Nautilus

(m) Cauliflower



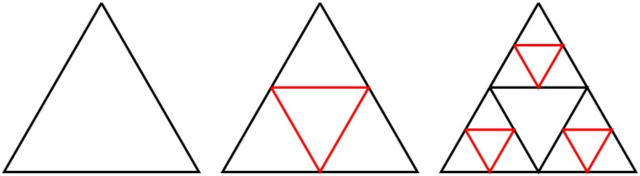
(n) coastline

The Sierpinski gasket

Given an equilateral triangle of vertices $\Gamma^0 = \{v_1, v_2, v_3\}$, consider the 3 maps $\psi_i : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

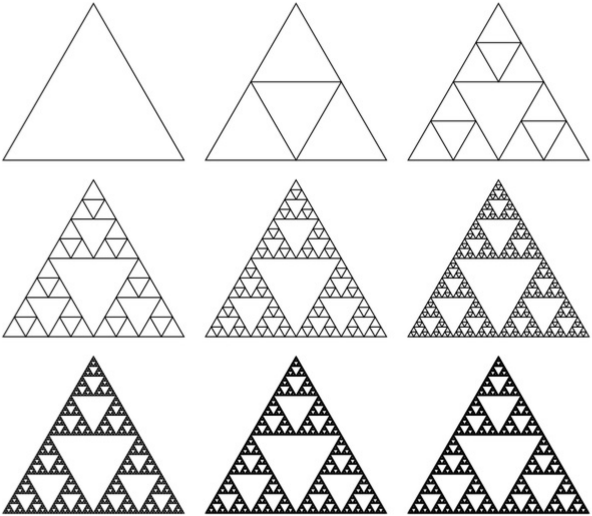
$$\psi_i(x) := v_i + \frac{1}{2}(x - v_i) \qquad x \in \mathbb{R}^2,\ i = 1, \dots, 3.$$

Iterating the ψ_i's, we get the set $\Gamma^\infty = \cup_{n=0}^{\infty} \Gamma^n$ where each prefractal Γ^n is given by the union of the images of Γ^0 under the action of the maps $\psi_{i_1} \circ \dots \circ \psi_{i_n}$ with $i_h \in \{1, \dots, 3\}$, $h = 1, \dots, n$.
The Sierpinski gasket S is the closure of Γ^∞ and it is the unique non empty compact set K which satisfies $K = \cup_{i=1}^{3} \psi_i(K)$.



(o) Γ^0 (p) Γ^1 (q) Γ^2

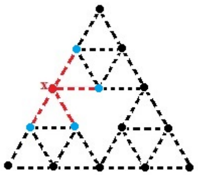
Sierpinski iterations



(r) Sierpinski iterations

The graph $(\Gamma^n, \sim)$

We consider a graph given by the vertices of Γ^n and the relation $x \sim y$ if and only if the segment connecting x and y is the image of a side of Γ under the action of some $\psi_{i_1} \circ \cdots \circ \psi_{i_n}$.



(s) Vertices in $\sim$ with $x \in \Gamma^2$

- $\partial\Gamma^n$, the boundary of the graph, is given by the vertices of Γ_0
- $\Gamma^n_{int} = \Gamma^n \setminus \Gamma^0$ is the set of the internal vertices

The Laplace operator on the Sierpinski gasket

To define the Laplacian on S, the idea is to introduce discrete operators on the $(\Gamma^n, \sim)$ and to perform a limit process.

- The probabilistic version of this approach was introduced by Kusuoka and Lindstrøm who considered suitable scaled random walks on Γ^n and then passed to the limit to define a Brownian motion on S.
- An analytical approach was taken by Kigami who considered scaled finite differences

$$\Delta^n f(x) = \sum_{y \sim x} \left(\frac{5}{3}\right)^n (f(y) - f(x))^2, \qquad x \in \Gamma^n$$

  and passed to the limit to define Δf on S.

The two approaches give rise to the same self-adjoint differential operator on S, the Laplacian Δ on S.

The eikonal operator on the Sierpinski gasket

To define the eikonal equation on S we will mimic the analytical approach of Kigami:

We consider graph eikonal equations on the prefractals and we show that the solutions of these problems converge to a function which satisfies an eikonal equation on the fractal set.

- The principle underlying the definition of Laplacian and harmonic functions on the Sierpinski gasket is the minimization of the energy.
- For the eikonal equation the principle is the optimal control interpretation of the solution at all the different levels (discrete on the prefractal and continuous on the fractal).

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The discrete eikonal equation on $(\Gamma^n, \sim)$

For a function $u : \Gamma^n \to \mathbb{R}$, we consider the discrete eikonal equation

$$\begin{cases} |Du(x)|_n = f(x) & \text{in } \Gamma^n_{int} \\ u = g & \text{on } \Gamma^0 \end{cases}$$

where

$$|Du(x)|_n := \max_{y \sim x} \left\{ -\frac{1}{h_n}(u(y) - u(x)) \right\},$$

$h_n = (1/2)^n$ and the function $f : S \to \mathbb{R}$ is continuous with

$$\lambda := \min_S f(x) > 0.$$

Remark:
The eikonal equation can be rewritten as

$$u(x) = \inf_{y \sim x} \{u(y) + h_n f(x)\}$$

Theorem

Let $g : \Gamma^0 \to \mathbb{R}$ be such that $g(x) \leq \inf \left\{ \sum_{k=0}^{N} h_n f(x_k) + g(y) \right\}$ $\forall x, y \in \Gamma^0$ where $x_i \sim x_{i+1}$, $i = 0, \ldots, N-1$ and $x_0 = x$, $x_N = y$.
Then, the **unique solution** of graph eikonal equation with Dirichlet boundary condition is given by the value function

$$u_n(x) = \min \left\{ \sum_{k=0}^{N} h_n f(x_k) + g(y) \right\}$$

where $x_i \sim x_{i+1}$, $i = 0, \ldots, N-1$ and $x_0 = x$, $x_N = y$.

Moreover

$$|u_n(x)| \leq \max_{\Gamma^0} |g| + d(x, \Gamma^0) \max_{\Gamma^n}\{f\} \qquad \forall x \in \Gamma^n$$

$$\frac{|u_n(y) - u_n(x)|}{h_n} \leq \max_{\Gamma^n}\{f\} \qquad \forall x, y \in \Gamma^n, x \sim y$$

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An immediate consequence of the previous result is that

the sequence $\{u_n\}_n$ uniformly converges to a function u on the Sierpinski gasket S

Following the the definition of Laplace equation on Sierpinski, it would be natural to say that u is the solution of the eikonal equation on S.

Is it possible to characterize u by an eikonal equation defined on S ?

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In Giga-Hamamuki-Nakayasu (Trans.AMS, 2015) is given a definition of metric viscosity solution for the eikonal equation in a general metric space $\mathcal{X}$.

To establish the link between the limit u of the sequence $\{u_n\}$ and the definition of metric viscosity solution we proceed in the following way

1. We define an appropriate notion of viscosity solution for continuous eikonal equations on the prefractal Γ^n
2. We estimate (uniformly in n) the distance between the solutions of the discrete and of the continuous eikonal equations on Γ^n
3. We show that the solutions of the eikonal equations on Γ^n are also metric viscosity solutions if $\mathcal{X} = \Gamma^n$.
4. By stability the sequence of the solutions of the continuous eikonal equation, and therefore also of discrete eikonal equations, on the prefractal Γ^n converges to the solution of the eikonal equation on S.

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Step 1: Eikonal equations on networks

Note that the study of Hamilton-Jacobi equation on networks is not a straightforward generalization of the Euclidean setting because of the presence of the vertices.

With the aim of extending the notion of viscosity solution to networks, several different approaches have been recently proposed. Here I will consider the definition in

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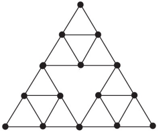
We consider the network S^n given by the prefractal $\Gamma^n = \{v_i\}_{i \in I}$ and by the arcs $E^n = \{e_j\}_{j \in J}$ connecting the vertices of Γ^n.



(t) Network S^2

We denote by

- $\pi_j : [0,1] \to \mathbb{R}$, a parametrization of the edge e_j
- $Inc_i := \{j \in J : \ x_i \in e_j\}$ the set of the arcs incident the vertex v_i
-

$$a_{ij} := \begin{cases} 1 & \text{if } x_i \in e_j \text{ and } \pi_j(0) = x_i, \\ -1 & \text{if } x_i \in e_j \text{ and } \pi_j(1) = x_i, \\ 0 & \text{otherwise.} \end{cases}$$

the signed incidence matrix $A = \{a_{ij}\}_{i \in I, j \in J}$ which gives the orientation of the edge induced by π_j.

Definition (test function)

Let $\varphi \in C^1(S^n)$.

i) Let $x \in e_j$, $j \in J$. We say that φ is test function at x, if the restriction the restriction of φ to e_j, i.e. $\varphi_j := \varphi \circ \pi_j$ is differentiable at $\pi_j^{-1}(x)$.

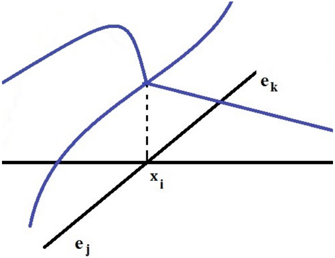
ii) Let $x_i \in \Gamma^n_{int}$, $j, k \in Inc_i$, $j \neq k$. We say that φ is (j,k)-test function at x, if φ_j and φ_k are differentiable at $\pi_j^{-1}(x)$ and $\pi_k^{-1}(x)$, with

$$a_{ij} D_j \varphi(\pi_j^{-1}(x)) + a_{ik} D_k \varphi(\pi_k^{-1}(x)) = 0,$$

where (a_{ij}) is the incidence matrix

Remark:

Condition $a_{ij} D_j \varphi(\pi_j^{-1}(x)) + a_{ik} D_k \varphi(\pi_k^{-1}(x)) = 0$ says that, taking into account the orientation, the function φ is differentiable at x_i along the direction given by the couple of edges e_j and e_k (no condition, except continuity, along the other incident edges)



(u) test function

Definition (network viscosity solution)

i) If $x \in e_j$, $j \in J$, then a function $u \in \mathrm{USC}(S^n)$ (resp., $v \in \mathrm{LSC}(S^n)$) is called a subsolution (resp. supersolution) at x if for any test function φ for which $u - \varphi$ attains a local maximum at x (resp., a local minimum), we have

$$|D_j\varphi(x)| \leq f(x) \qquad \big(\text{resp.,} \quad |D_j\varphi(x)| \geq f(x)\big);$$

ii) If $x = x_i \in \Gamma^n_{int}$, then

- A function $u \in \mathrm{USC}(S^n)$ is called a subsolution at x if for any $j, k \in Inc_i$ and any (j,k)-test function φ for which $u - \varphi$ attains a local maximum at x relatively to $e_j \cup e_k$, we have $|D_j\varphi(x)| \leq f(x)$.
- A function $v \in \mathrm{LSC}(S^n)$ is called a supersolution at x if for any $j \in Inc_i$, there exists $k \in Inc_i \setminus \{j\}$ (said feasible for j at x) such that for any (j,k)-test function φ for which $u - \varphi$ attains a local minimum at x relatively to $e_j \cup e_k$, we have $|D_j\varphi(x)| \geq f(x)$.

Remark: Note that the definitions of sub and supersolution at the vertices are not symmetric.

Theorem

Let $g : \Gamma \to \mathbb{R}$ be such that $g(x) \leq \inf_\xi \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \; \forall x, y \in \Gamma^0$ where ξ is a piecewise differentiable path such that $\xi(0) = x$, $\xi(T) = y$. Then the unique network viscosity solution u_n of the eikonal equation on S^n with the boundary condition $u = g$ on Γ^0 is given by

$$u_n(x) = \inf_\xi \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \quad \text{for all } x \in S^n.$$

Moreover, u is bounded and Lipschitz continuous and

$$|u_n(x)| \leq \max_{\Gamma^0} |g| + \max_{S^n} |f| d(x, \Gamma^0) \quad \text{for all } x \in S^n,$$

$$\|Du_n\|_\infty \leq \max_{S^n} |f|,$$

the sequence $\{u_n\}_n$ is decreasing.

Step 2: Estimate between the solutions of discrete and continuous eikonal equation on networks

Proposition

Let u_{h_n} and u_n be respectively the solutions of discrete and continuous eikonal equations on the prefractal Γ^n. Then

$$|u_n(x) - u_{h_n}(x)| \leq C\omega_f(h_n^{1/2}) \qquad \forall x \in \Gamma^n$$

where ω_f is the modulus of continuity of f. Hence the sequences u_{h_n} and u_n converge to the same limit u on S.

The proof is based on the classical doubling of variables argument in viscosity solution theory adapted to networks

$$\Psi(x, y) = u_{h_n}(x) - u_n(y) - \frac{d(x, y)^2}{2\varepsilon}$$

Step 3: Giga-Hamamuki-Nakayasu metric viscosity solutions

• Given a path $\xi : I \subset \mathbb{R} \to \mathcal{X}$, define the **metric derivative** of $|\xi|$ by

$$|\xi'|(t) := \lim_{s\to t} \frac{d_{\mathcal{X}}(\xi(s),\xi(t))}{|s-t|}.$$

(note in general $\xi'(t)$ may be not well defined).

Let $\mathcal{A}_x(I)$ be the set of absolutely continuous curves with $\xi(0) = x$.

• A function u is said arcwise upper (resp., lower) semicontinuous if for each $\xi \in \mathcal{A}(I)$ the function $u \circ \xi$ is upper (lower) semicontinuous in I.

• For a function $w : \mathbb{R} \to \mathbb{R}$, denote by $D^{\pm}w(t)$ respectively its super- and subdifferential at the point t.

In a Euclidean space we have

$$|Du(x)| = \sup_{\xi:|\xi'(0)|\leq 1} |(u \circ \xi)'(0)|$$

Hence reflecting this property we have the following definition

Subsolution

Let $\Omega \subset \mathcal{X}$. An arc-wise u.s.c. function u is said a subsolution if for each $x \in \Omega$ and for all $\xi \in \mathcal{A}_x(I, \Omega)$ with $|\xi'| \leq 1$, the function $u \circ \xi : I \subset \mathbb{R} \to \mathbb{R}$ satisfies

$$|p| \leq f(x) \qquad \forall p \in D^+ u(\xi(0)).$$

The definition of a supersolution is more involved. Roughly speaking, v is a **supersolution** if, fixed x, for any $\varepsilon > 0$ there is a curve ξ with $\xi(0) = x$ and $|\xi'| \leq 1$ such that $|(v \circ \xi)'(t)| \geq f(\xi(t)) - \varepsilon$ for all t until ξ hits the boundary (superoptimality principle).
Define the exit and entrance time of ξ in Ω by

$$T^{+}_{\Omega}[\xi] := \inf\{t \in [0, +\infty) : \xi(t) \in \partial\Omega\} \in [0, +\infty]$$
$$T^{-}_{\Omega}[\xi] := \sup\{t \in (-\infty, 0] : \xi(t) \in \partial\Omega\} \in [-\infty, 0].$$

Supersolution

An arc-wise l.s.c. $v \in \mathrm{LSC}(\Omega)$ is said a supersolution if for each $x \in \Omega$ and $\varepsilon > 0$, there exists $\xi \in \mathcal{A}_x(\mathbb{R}, \Omega)$ and $w \in \mathrm{LSC}(\mathbb{R})$ such that

$$\begin{cases} T^{\pm} := T^{\pm}_{\Omega}[\xi] \text{ are both finite,} \\ w(0) = v(x), \quad w(t) \geq v(\xi(t)) - \varepsilon \quad \forall t \in (T^{-}, T^{+}), \\ |p| \geq f(x) - \varepsilon \qquad \forall p \in D^{-}w(t),\ t \in (T^{-}, T^{+}). \end{cases}$$

Remarks:

• The main point is that the previous definitions are 1-dimensional and they do not involve any definition of gradient on $\mathcal{X}$.

• The definition of metric viscosity solution is consistent with the classical one in the Euclidean space.

• The definition of subsolution and supersolution are not symmetric.

• The definition of supersolution is not local.

Prop. (existence/uniqueness on S)

Let S the Sierpinski gasket and let $g : \Gamma \to \mathbb{R}$ be such that $g(x) \le \inf_\xi \left\{ \int_0^T f(\xi(t))dt + g(y) \right\}$ for all $x, y \in \Gamma^0$ where $\xi \in \mathcal{A}((0,T), S)$ such that $\xi(0) = x$, $\xi(T) = y$.
Then the unique metric viscosity solution of

$$\begin{cases} |Du| = f(x) & x \in S \\ u = g & x \in \Gamma^0 \end{cases}$$

is given by

$$u(x) = \inf \left\{ \int_0^T f(\xi(t))dt + g(y) \right\} \quad \text{for all } x \in S$$

where $y \in \Gamma$ and $\xi \in \mathcal{A}((0,T), S)$ such that $\xi(0) = x$, $\xi(T) = y$.

Moreover u is continuous and bounded.

Step 4: Passage to the limit

Equivalence on the prefractal S^n

A function $u \in \mathrm{USC}(S^n)$ (resp. $v \in \mathrm{LSC}(S^n)$) is a metric viscosity subsolution (resp. supersolution) if and only if it is a network viscosity subsolution (resp. supersolution)

Remark:

• On a network arcwise continuity and continuity coincide.

• The main difficult is to show the equivalence between the (nonlocal) definition of metric viscosity supersolution and the one of network viscosity supersolution.

Theorem (Convergence)

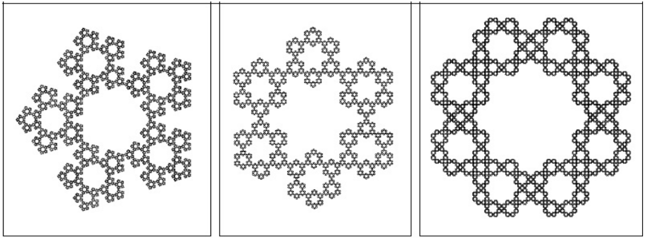
Let u_n be the sequence of the network viscosity solutions of eikonal equation on the network S^n. Then, as $n \to +\infty$, u_n tends to u uniformly in S where u is the metric viscosity solution of

$$\begin{cases} |Du| = f(x) & x \in S \\ u = g & x \in \Gamma^0 \end{cases}$$

Hence, as $n \to +\infty$, the sequence $\{u_{h_n}\}_n$ of the solutions of graph eikonal equation also uniformly converges to u.

Remark:
The proof is based on stability properties of the metric viscosity solutions and the equivalence on Γ^n of metric and network viscosity solutions.

- **More general fractals:** The interior approximation method can be extended to a class of post-critically finite self similar sets (generally speaking, these sets are obtained by subdividing the initial cell into cells of smaller and smaller size and the cells must intersect at isolated points) for which the corresponding prefractals are expanding, i.e. $\Gamma_n \subset \Gamma_{n+1}\ \forall n \in \mathbb{N}$.



(v) post critically finite fractals

- **More general Hamiltonians:** Whereas the notion of metric viscosity solution is restricted to the case of the eikonal equation $|Du| = f(x)$, the results on prefractals can be extended to the Hamilton-Jacobi equation $H(x, Du) = 0$ where the Hamiltonian $H(x, p)$ is convex and coercive with respect to p. Hence we obtain the sequence of the solutions of the Hamilton-Jacobi equation on the prefractal converges uniformly to a function u defined on the Sierpinski gasket.
  At present since there is no definition of viscosity solution for the equation $H(x, Du) = 0$ on S for a general Hamiltonian, this can be seen as a constructive way to define Hamilton-Jacobi equations on the Sierpinski gasket.
- **Boundary conditions:** Following the classical definition of Laplacian on the Sierpinski gasket, we imposed the boundary conditions on the vertices of Γ^0. By easy modifications, it is possible to consider the problem in a connected subdomain Ω of S imposing the boundary condition on any finite subset of vertices contained in Ω.

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Thank You!