A variational approach to fully nonlinear operators.

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per MAURIZIO FALCONE.

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Let Ω be a bounded smooth domain of $\mathbb{R}^N$. Let

$$\lambda_1 := \inf_{u\in H_o^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

it is well known that the infimum is achieved by the Dirichlet eigenfunction of $-\Delta$, i.e. by φ a solution of

$$\begin{cases} \Delta\varphi + \lambda_1\varphi = 0 & \text{in } \Omega \\ \varphi > 0 \text{ in } \Omega & \varphi = 0 \text{ on } \partial\Omega. \end{cases} \qquad (1)$$

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Another characterisation of λ_1 is given by

$$\lambda_1 = \sup\{\mu \in \mathbb{R}, \ \text{ such that } \exists \psi > 0, \Delta\psi + \mu\psi \le 0 \text{ in } \Omega\}.$$

The advantage of this definition of

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The disadvantage of this definition is that it cannot be used to compute it through a numerical approach.

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was introduced by Berestycki, Nirenberg and Varadhan (1993) for $Lu = \mathrm{tr}(A(x)D^2u)$ with

$$\lambda I \leq A(x) \leq \Lambda I$$

and no regularity required on $\partial\Omega$.

This definition of eigenvalue

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It has been generalised to fullynonlinear operators (Busca-Esteban-Quaas, Ishii-Yoshimura, Birindelli-Demengel)

- Hypothesis
  $\lambda \operatorname{tr} N \leq F(x,p,M+N) - F(x,p,M) \leq \Lambda \operatorname{tr} N$
  for $0 < \lambda \leq \Lambda$ and $N \geq 0$.

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For other generalizations see also Berestycki-Rossi, Berestycki-Capuzzo Dolcetta-Porretta-Rossi.

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is an "eigenvalue" that corresponds to an "eigenfunction" $\phi > 0$ in Ω

$$\begin{cases} F(x,\nabla\phi, D^2\phi) + \lambda_1\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

The main example of fully operator we wish to consider are the so call Pucci operators

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) = \inf_{\lambda I \le A \le \Lambda I} tr(AD^2u).$$

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These extremal operators play a very important role in the theory of fully nonlinear operators since any elliptic operator F $(\lambda \operatorname{tr} N \leq F(x, M+N) - F(x, M) \leq \Lambda \operatorname{tr} N)$, satisfies

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) \leq F(x, D^2u) \leq \mathcal{M}^{+}_{\lambda,\Lambda}(D^2u).$$

Hence solutions of $F(x, D^2u) = f$, are sub or super solutions for the extremal operators.

Find some sort of "Rayleigh quotient" to characterize the eigenvalues for non variational operators, be they linear or fully nonlinear.

The idea being to approximate non variational operators with variational non local operators.
This is a joint work in progress with L. Caffarelli and S. Patrizi.

It is well known that the Rayleigh quotient allows to use Numerical schemes in order to compute the eigenvalues and eigenfunctions of the Laplacian. I will just mention the work of

1. **Babuska, I.; Osborn, J.** Eigenvalue problems, monography in *Handbook of numerical analysis*, 1991.
2. **Babuska, I.; Osborn, J.** E. Finite element-Galerkin approximation of the eigenvalues and eigenvectors of self-adjoint problems. *Math. Comp.* 52 (1989).

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$$\varphi_\epsilon^M(y) = \frac{1}{\epsilon^n \det(M)} \varphi\left(M^{-1} \frac{y}{\epsilon}\right).$$

Proposition (Caffarelli-Silvestre)

If $u \in C^2(\Omega)$ then, for ϵ going to zero

$$I_\epsilon^M[u](x) := \int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{\epsilon^2} \varphi_\epsilon^M dy \to Lu = \mathrm{tr}(M^2 D^2 u).$$

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Taylor approximation

$$= \frac{1}{2\epsilon^2} \int_{\mathbb{R}^N} [\langle D^2 u(x) \epsilon M\eta, \epsilon M\eta \rangle + o(\epsilon^2)] \varphi(\eta) d\eta$$

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Now since $\int_{\mathbb{R}^N} \eta_i\eta_j\varphi(\eta)d\eta = 2\delta_{ij}$,

$$\frac{1}{2} \int_{\mathbb{R}^N} \langle MD^2 u(x)M\eta, \eta\rangle \varphi(\eta)d\eta = \mathrm{tr}(M^2 D^2 u).$$

For $M \in \mathcal{A}$ constant, let $E_\epsilon^M : H_0^1(\Omega) \to \mathbb{R}$ be the energy:

$$E_\epsilon^M[u] = \frac{\lambda}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{\epsilon^2} \varphi_\epsilon^M(x-y)dydx,$$

where $u \in H_0^1(\Omega)$ is extended to zero outside Ω.

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$$\lambda_1^{\epsilon,M} := \inf_{u \in H_0^1(\Omega)} \frac{E_\epsilon^M[u]}{\int_\Omega u^2 dx}. \tag{2}$$

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Theorem (B.-Caffarelli-Patrizi)

The infimum is achieved by $\varphi \in H_0^1(\Omega)$ solution of

$$P_\epsilon^M[\varphi] + \lambda_1^{\epsilon,M}\varphi = 0 \quad \text{in } \Omega,$$

where $P_\epsilon^M[u] := \frac{\lambda}{2}\Delta u + \int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{\epsilon^2}\varphi_\epsilon^M(y)dy$. Furthermore $\varphi > 0$ in Ω so $\lambda_1^{\epsilon,M}$ is the principal eigenvalue for $-P_\epsilon^M$ in Ω.

Is it possible to use the *Finite element-Galerkin approximation* method given by Babuska and Osborne to compute $\lambda_1^{\epsilon,M}$?

In the fully nonlinear case we consider the energy

$$E_\epsilon[u] = \frac{\lambda}{2}\int_\Omega |\nabla u|^2 dx + \sup_{M\in\mathcal{A}} \frac{1}{2}\int_\Omega\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{\epsilon^2}\varphi_\epsilon^M(x-y)dydx$$

recall that $\mathcal{A} = \{M \in S(n),\ \frac{\lambda}{2} I \le M^2 \le \left(\Lambda - \frac{\lambda}{2}\right) I\ \}$.

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$$P_\epsilon[\varphi] + \lambda_1^\epsilon \varphi = 0 \quad in\ \Omega,$$

where

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and u be a solution of

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Theorem (Caffarelli-Silvestre)

Assume that $g \in C^1$, f is Lipschitz continuous and Ω satisfies the exterior ball condition. There exists $C > 0$ and $\alpha > 0$ such that

$$\|u_\epsilon - u\|_\infty \le C\epsilon^\alpha \left(\|g\|_{C^1} + \|f\|_{Lip}\right).$$

Since E_ϵ^M is coercive and, by convexity, lower semicontinuous with respect to the norm of $H_0^1(\Omega)$, $\lambda_1^{\epsilon,M}$ is a minimum of E_ϵ^M and any minimizing function is a solution of the Euler-Lagrange equation; precisely, for any $v \in C_c^\infty(\Omega)$

$$
\begin{aligned}
& \frac{\lambda}{2}\int_\Omega \nabla u \cdot \nabla v dx + \\
+ \quad & \frac{1}{2}\int_\Omega \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{\epsilon^2}\varphi_\epsilon^M(x-y)dydx \\
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We just need to prove that u is positive in Ω.

Theorem (Caffarelli-Silvestre)

Let f and g be continuous functions and Ω be a smooth domain then there exists $u \in C^{1,\alpha}(\Omega)$ solution of

$$\begin{cases} P_\epsilon^M[u] = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Furthermore, the maximum principle holds i.e. if $f \leq 0$ and $g \geq 0$ then $u \geq 0$ in Ω.

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Proposition (Strong comparison for eigenfunction, BCP)

Let u and v be respectively bounded sub and supersolution of

$$\begin{cases} P_\epsilon^M[u] + \lambda u = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \Omega^c. \end{cases}$$

If $v > 0$ in Ω and $u(x_0) > 0$ for some $x_0 \in \Omega$, then there exists $t > 0$ such that $v \equiv tu$.

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lax Prin.) By Krein Rutman theorem there exists λ^{KR} and $u_o > 0$ in Ω such that

$$P^M_\epsilon[u_o] + \lambda^{KR} u_o = 0 \quad \text{in} \quad \Omega, u_o = 0 \quad \text{on} \quad \Omega^c.$$

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(Definit.) Multiplying by u_o and integrating by parts, we get $\lambda^{KR} = \frac{E_\epsilon^M[u_o]}{\int_\Omega u_o^2 dx}$, therefore $\lambda_1^{\epsilon,M} \leq \lambda^{KR}$.

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(S.C.Pr.) Consequently $u_o > 0$ satisfies $P_\epsilon^M[u_o] + \lambda_1^{\epsilon,M} u_o \leq 0$, while u is somewhere positive. Hence we can apply the strong comparison Proposition and $u_o \equiv tu$.

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(End) $u > 0$ in Ω.

The non linear case requires a few more steps, and uses the linear case result.

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  Considering the Rayleigh quotient would give eigenvalues for a general class of quasi linear operators.
- Non linear Liouville's theorems for $P_\epsilon^M[u] + u^p = 0$ i.e. non existence of entire positive solution for some range of p.

Tanti auguri MAURIZIO

Theorem (Krein-Rutman)

Let X be Banach space and $G \subset X$ a closed and convex cone with vertex at the origin. If T is a compact linear mapping such that $T(G) \subset G$ and such that there exist $e \neq 0$ in G and a positive constant ρ satisfying

$$Te - \rho e \in G \tag{3}$$

then $r(T) := \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}} > 0$.
Thus in particular $\mu_o = r(T)$ is an eigenvalue for T and there exists $u_o \in G$ such that $T(u_o) = \mu_o u_o$.