On the coupling between MPC and DP methods for optimal control problems.

Alessandro Alla

work in collaboration with G. Fabrini and M. Falcone

UHH

Happy Birthday Prof

Rome, December 5, 2014

Outline

1. Background results
   - Model Predictive Control
   - Hamilton-Jacobi-Bellman equations
2. Coupling MPC with Bellman Equations
3. Numerical Tests

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Model Problem

Dynamics:

$$\dot{y}(t,u) = f(y(t),u(t)), y(t_0) = x \quad t \in (t_0,\infty]$$

Cost functional:

$$J_x(u) = \int_{t_0}^{\infty} L(y_x(s,u),u(s))e^{-\lambda s}\,ds, \quad \lambda \in \mathbb{R}^+$$

Value Function, DPP and HJB

$$v(x) := \inf_{u(\cdot)\in\mathcal{U}} J_x(y_x,u(\cdot)),$$

$$v(x) = \min_{u\in\mathcal{U}} \left\{ \int_{t_0}^{t} L(y_x(s,u),u(s))\,ds + v(y_x(t;u))e^{-\lambda s} \right\},$$

$$\lambda v(x) - \min_{u\in U}\{f(x,u)\cdot Dv(x) + L(x,u)\} = 0.$$

Re: Tesi Roma-Graz

Posta in arrivo x | alla2004@alice.it x | UNIVERSITA' x

Maurizio Falcone <falcone@mat.uniroma1.it> 22/04/08

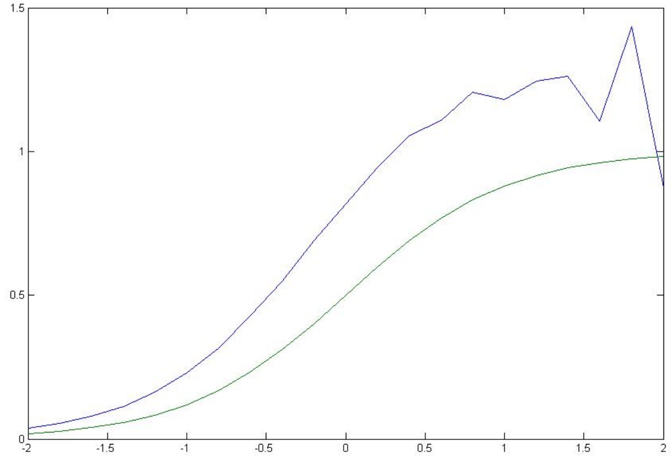
a alla2004

Mi dispiace deluderti ma in Europa si parla Inglese solo in Inghilterra e Irlanda ed il livello delle universita' li' e' piu' basso che da noi (salvo Oxford e Cambridge).
Dunque o impari l'inglese o impari la matematica.
Mi sembra che Graz sia un ottimo posto e la proposta di una tesi in comune mi sembra decisamente interessante.
Per imparare l'inglese esistono mille modi (corsi estivi, periodi all'estero di varia natura, corsi in italia,...) non credo sia questo il problema piu' urgente.
Ciao,

maurizio

Figure: 2009: Master Thesis (left), 2014: PhD Thesis (right).

Numerical Simulation in my Bachelor's Thesis



Thanks to Simone Cacace

Model Predictive Control

Discrete dynamics:

$$\begin{cases} y(t_{n+1}, u(t_n)) = F(y(t_n), u(t_n)), & t \in (t_0, \infty], \\ y(t_0) = x, \end{cases}$$

Discrete Cost Functional:

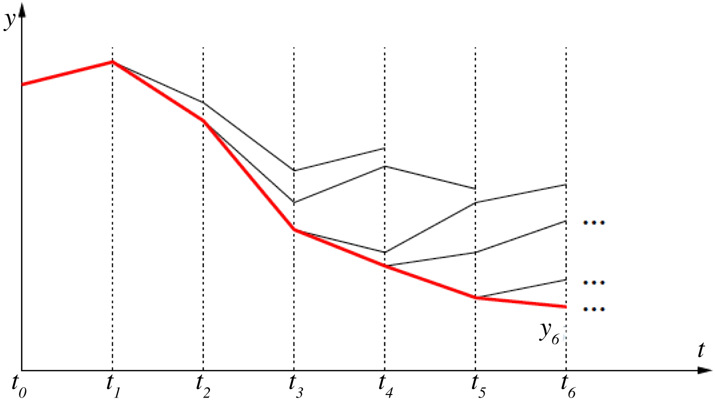
$$J_x(y(t_n), u(t_n)) = \sum_{n=0}^{\infty} L(y(t_n; u), u(t_n))$$

Finite Horizon Problem:

$$\min_{u \in U} J^N_{y(n)}(y, u) = \sum_{k=0}^{N-1} L\left(y(t_{n+k}, u(t_{n+k}))\right)$$

NO TERMINAL CONSTRAINTS, NO DISCOUNT FACTOR

IDEA of MPC (thanks to Prof. Grüne)



Philosophical Question!

What's the smallest N which ensures (asymptotic) Stability?
Relaxed Dynamic Programming Principle

Bellman's Equation

Dynamic Programming Principle:

$$v(x) = \min_{u \in \mathcal{U}} \left\{ \int_{t_0}^{t} L(y_x(s,u), u(s))\, ds + v(y_x(t;u)) e^{-\lambda s} \right\}.$$

Discrete Approximation (Value Iteration)

$$V_i^{k+1} = \min_{u \in U} \{ e^{-\lambda \Delta t} I \left[V^k \right] (x_i + \Delta t\, f(x_i, u)) + \Delta t\, L(x_i, u) \}$$

This algorithm converges for any initial guess V^0.

Error Estimate:

$$\max_{i \in N_G} \| v(x_i) - V_i \| \leq C(\Delta t)^{1/2} + \frac{L_f}{\lambda(\lambda - L_f)} \frac{\Delta x}{\Delta t}.$$

N_G : number of nodes, L_f : Lipschitz constant of the dynamic f.

Advantages and Disadvantages of the methods

DP's PROS

1. Knowledge Value Function.
2. A-priori error estimates in L^∞
3. General approach
4. Computation of feedbacks

MPC's PROS

1. Easy implementation
2. Short computational time
3. High dimensional problems

DP's CONS

The "curse of dimensionality"

1. computational cost
2. huge memory allocations.

MPC's CONS

1. Feedback along one traj.
2. Selected problems

Outline

MPC–HJB Algorithm

Algorithm

Start: Inizialization

Step 1: Solve MPC and compute y_x^{MPC} for a given initial condition x

Step 2: Compute the distance from y_x^{MPC} via the Eikonal equation

Step 3: Select the tube Ω_ρ with distance ρ with respect to y_x^{MPC}

Step 4: Compute the constrained value function v^{tube} in Ω_ρ via HJB

Step 5: Compute the optimal feedbacks and trajectory using v^{tube}.

End

Outline

Infinite Horizon Problem for the Van der Pol dynamics

Dynamics:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = (1 - x(t)^2)y(t) - x(t) + u(t) \\ x(0) = x_0, \; y(0) = y_0. \end{cases}$$

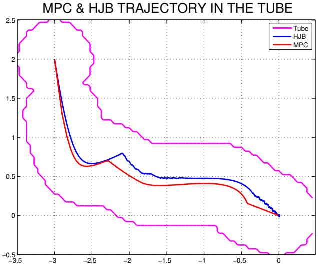
Cost Functional

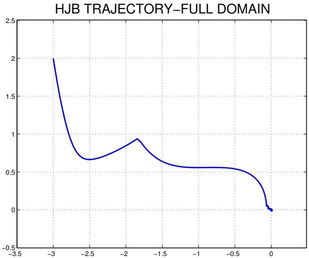
$$J_x(u) := \int_0^{\infty} (x^2 + y^2) e^{-\lambda t} \, dt.$$

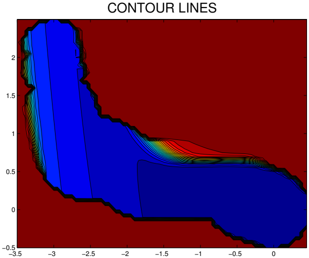
Parameters

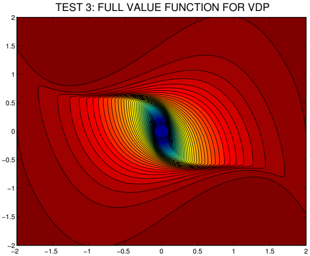
$\lambda = \{0.1, 1\}, u \in [-1, 1]$, #contr value $= 21$, #contr traj $= 3$, $\rho = 0.4, \Omega = [-6, 6]^2, \Delta t_{MPC} = 0.05 = \Delta t_{HJB}, \Delta x_{HJB} = 0.025$, $x(0) = -3, \; y(0) = 2$.

Infinite Horizon Problem for the Van der Pol dynamics









Infinite Horizon Problem for the Van der Pol dynamics

$\lambda = 0.1$	**MPC N=10**	**HJB in Ω_ρ**	**HJB in Ω**
CPU	65s	112s	152s
$J_x(u)$	14.31	13.13	12.41

$\lambda = 1$	**MPC N=10**	**HJB in Ω_ρ**	**HJB in Ω**
CPU	11s	27s	46s
$J_x(u)$	6.45	6.12	6.07

Minimum time problem for Zermelo dyanmics

Dynamics:

$$\begin{cases} \dot{x}(t) = 1 + V_b \cos(u(t)), & V_b \in \mathbb{R} \\ \dot{y}(t) = V_b \sin(u(t)) & t \in [0, t_f). \end{cases}$$

Cost Functional

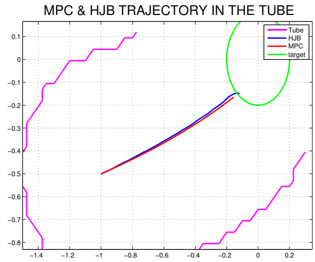
$$J_x(u) := \int_0^{t_f} \ell(x, u)\chi_{\mathcal{T}}(t)\, e^{-\lambda t}, dt$$

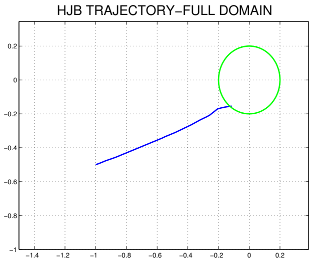
with $\mathcal{T} = B_\varepsilon(0)$, $\varepsilon = 0.2$, running cost $\ell(x, a) = \|x\|^2$

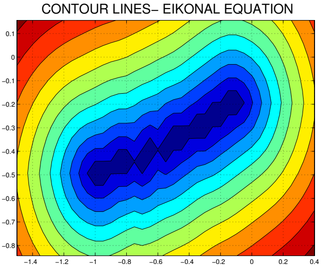
Parameters:

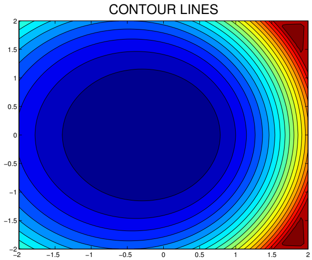
$\lambda = 0.1$, $V_b = \{0.6, 1.4\}$, $U = [-\pi, \pi]$, $\#$contr value $= 72 = \#$contr traj, $\rho = 0.4$, $\Omega = [-2, 2]^2$, $\Delta t_{MPC} = 0.05 = \Delta t_{HJB}$, $\Delta x_{HJB} = 0.04$, $[T_0, T] = [0, 1]$, $x_0 = [-1, -0.5]$.

Minimum time problem for Zermelo dyanmics









Minimum time problem for Zermelo dyanmics

$V_b = 0.6$	**MPC**	**HJB in Ω_ρ**	**HJB in Ω**
CPU	1.36s	11.15s	24.47s
FV	0.35	0.34	0.34

$V_b = 1.4$	**MPC**	**HJB in Ω_ρ**	**HJB in Ω**
CPU	0.92s	6.39s	17.61s
FV	0.23	0.2	0.2

THANK YOU FOR YOUR ATTENTION!!!

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