UN MODELLO VARIAZIONALE PER LA RICOSTRUZIONE DI IMMAGINI CON CONTORNI NASCOSTI

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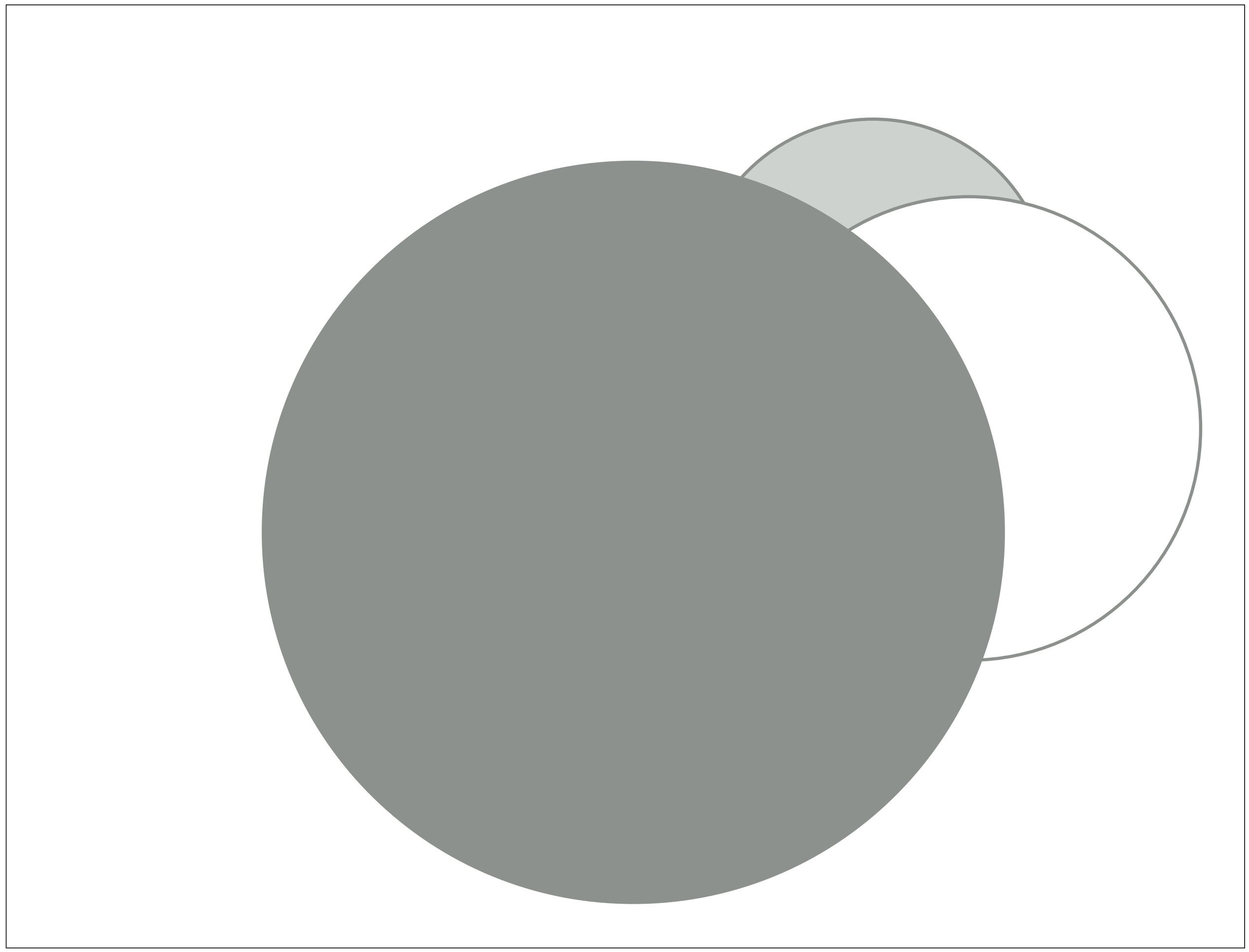
SHAPES WITH OCCLUSIONS



OVERLAPPING PARTITIONS OF $\mathbb{R}^2$

Model for image segmentation that incorporates (partially) the way that an image g derives from a 2D projection of a 3D scene.

Structure of the model:

(i) a collection of overlapping sets with finite perimeter

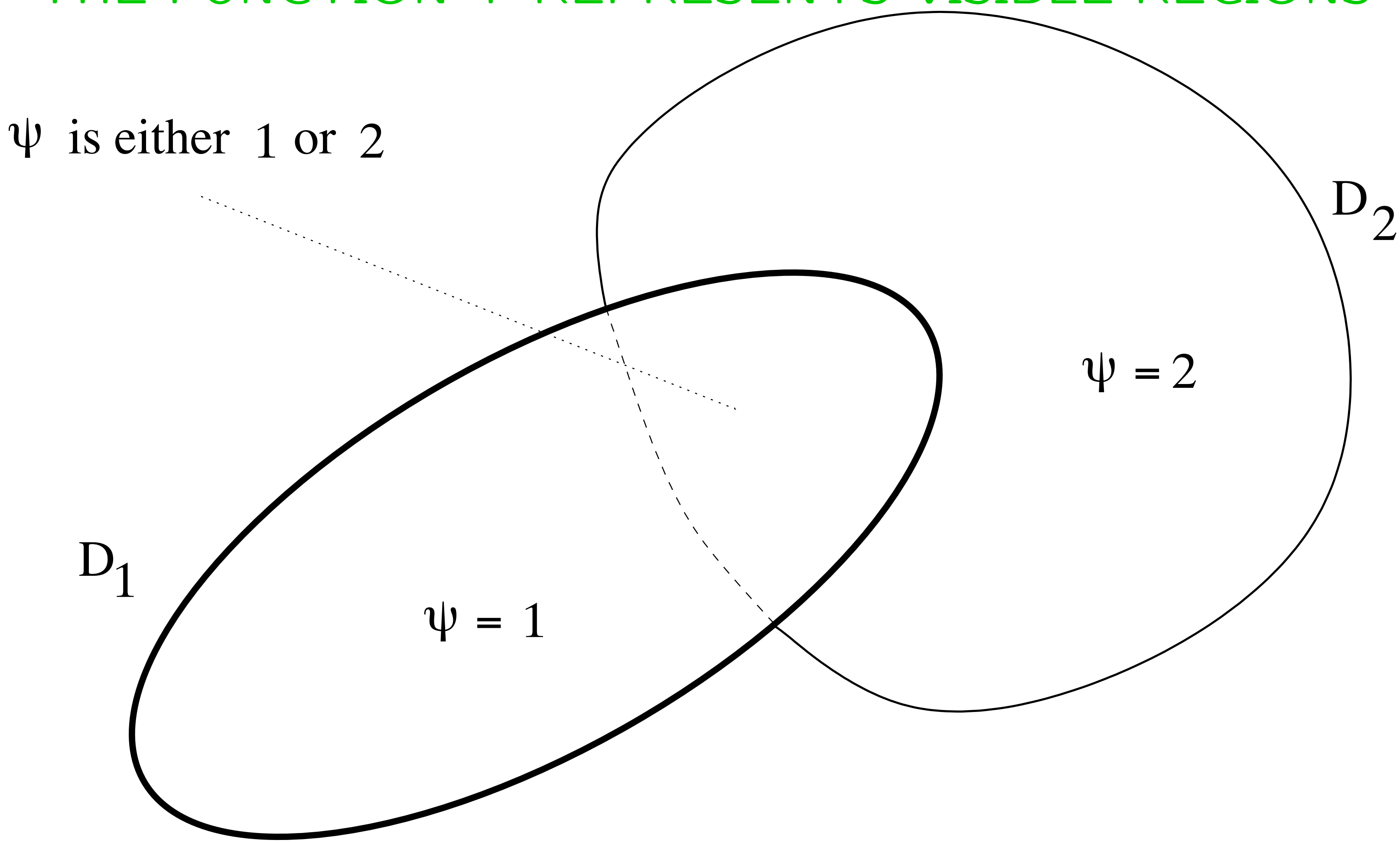
$$\{E_1, \ldots, E_n\}, \qquad E_i \subset \mathbb{R}^2 \ \forall i$$

(ii) a function $\psi \in BV_{\text{loc}}(\mathbb{R}^2; \mathbb{N})$ with integer values defined by

$$\begin{cases} \psi(x) \in \Big\{ i \in \{1, \ldots, n\} : x \in E_i \Big\} \\ \psi(x) = 0 \quad \text{if } x \notin \cup_{i=1}^n E_i. \end{cases}$$

The set $\{x \in \mathbb{R}^2 : \psi(x) = i\}$ represents the visible part of E_i.

THE FUNCTION ψ REPRESENTS VISIBLE REGIONS



A FUNCTIONAL FOR IMAGE SEGMENTATION WITH OCCLUSIONS

(iii) a function $u \in SBV_{\text{loc}}(\mathbb{R}^2)$ with $\nabla u \equiv 0$, as in piecewise constant Mumford-Shah segmentation, such that the inclusion $J_u \subseteq J_\psi$ between the sets of jumps holds.

We define the functional

$$\mathcal{G}_\lambda(u, n, \{E_1, \ldots, E_n\}, \psi) = \lambda \int_{\mathbb{R}^2} (u-g)^2 dx + \sum_{i=1}^{n} \mathcal{F}(E_i)$$

where $g \in L^\infty(\Omega)$ with compact support is the input image.

$\mathcal{F}(E_i)$ is a curvature depending functional that is added to an energy of the Mumford-Shah type.

CURVATURE DEPENDING PART

$$\mathcal{G}_\lambda(u, n, \{E_1, \ldots, E_n\}, \psi) = \int_{\mathbb{R}^2} (u-g)^2 dx + \sum_{i=1}^{n} \mathcal{F}(E_i)$$

$\mathcal{F}(E_i)$ is a curvature depending functional:

$$\mathcal{F}(E_i) := \int_{\partial E_i} [1 + |\kappa_i(x)|^p] d\mathcal{H}^1(x)$$

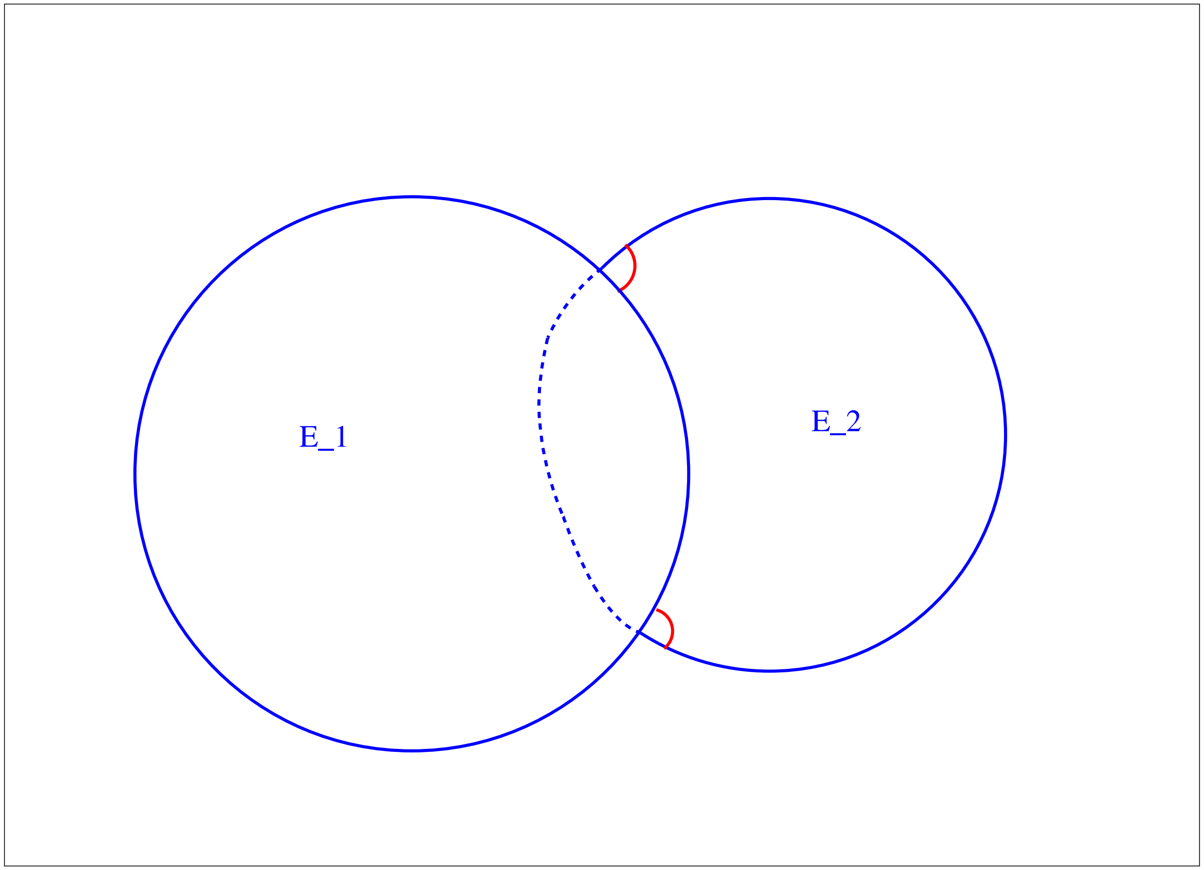
$\kappa_i(x)$ is the curvature of ∂E_i at x, $p > 1$.

The functional $\mathcal{G}$ is defined on the domain

$$\mathcal{D} = \left\{(u, n, \{E_1, \ldots, E_n\}, \psi) : J_u \subseteq J_\psi \subseteq \cup_{i=1}^{n} \partial^* E_i\right\}$$

EFFECT OF CURVATURE

$$\mathcal{F}(E_i) = \int_{\partial E_i} [1 + |\kappa_i|^2] d\mathcal{H}^1$$



SAMPLE OF BIBLIOGRAPHY

Nitzberg and Mumford (1990)

Nitzberg, Mumford and Shiota (1993)

Bellettini and Paolini (1995)

Masnou (2002)

Esedoglu and Shen (2002)

Ballester, Caselles and Verdera (2003)

Bellettini and M. (2004)

Masnou and Morel (2006)

Cao, Gousseau, Masnou and Pérez (2011)

THE NITZBERG-MUMFORD-SHIOTA FUNCTIONAL

Partial ordering of sets E_i that represents relative depth

if $i < j$ then E_i occludes E_j

The visible part of the region E_i is the set E_i':

$$E_1' = E_1, \qquad E_i' = E_i \setminus \bigcup_{j=1}^{i-1} E_j$$

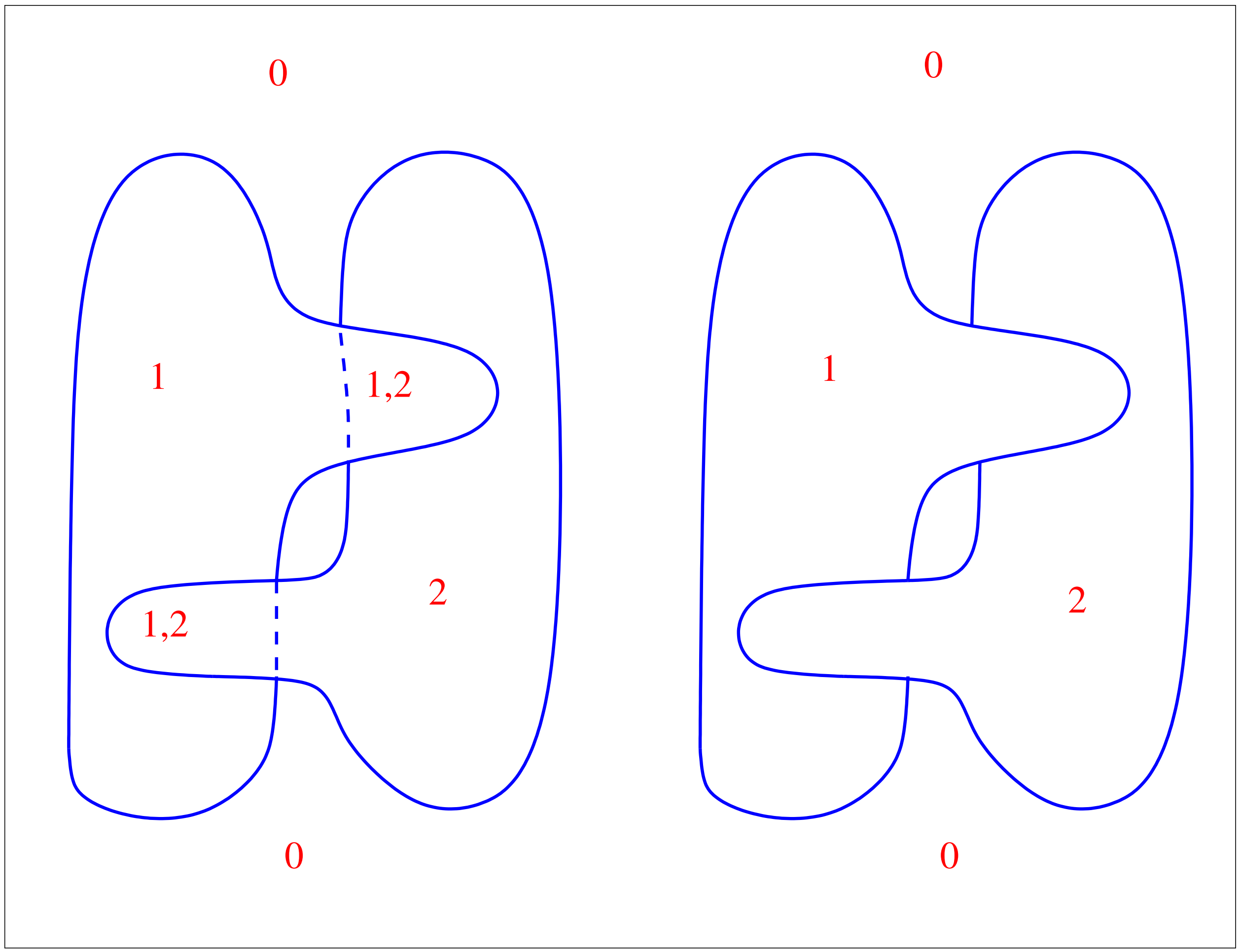
Energy of an ordered family of overlapping regions:

$$\mathcal{G}_\lambda^0(E_1, \ldots, E_n) = \lambda \sum_{i=1}^{n+1} \int_{E_i'} (c_i - g)^2 dx + \sum_{i=1}^{n} \mathcal{F}(E_i)$$

c_i are constants and E_{n+1}' is the background region.

Interwoven shapes are not allowed.

VALUES OF Ψ FOR INTERWOVEN SHAPES



A SIMPLE, SPECIFIC INSTANCE OF g FUNCTION

Let $M \in \mathbb{N}$ and $\{p_1, \ldots, p_{2M}\} \subset \mathbb{R}^2$; let $\{D_1, D_2\} \subset \mathcal{C}^2(\mathbb{R}^2)$ be connected sets with the following properties (not interwoven):

$$D_1 \cap D_2 \neq \emptyset, \qquad \partial D_1 \cap \partial D_2 = \{p_1, \ldots, p_{2M}\}.$$

∂D_1 and ∂D_2 intersect transversally at $\{p_1, \ldots, p_{2M}\}$,

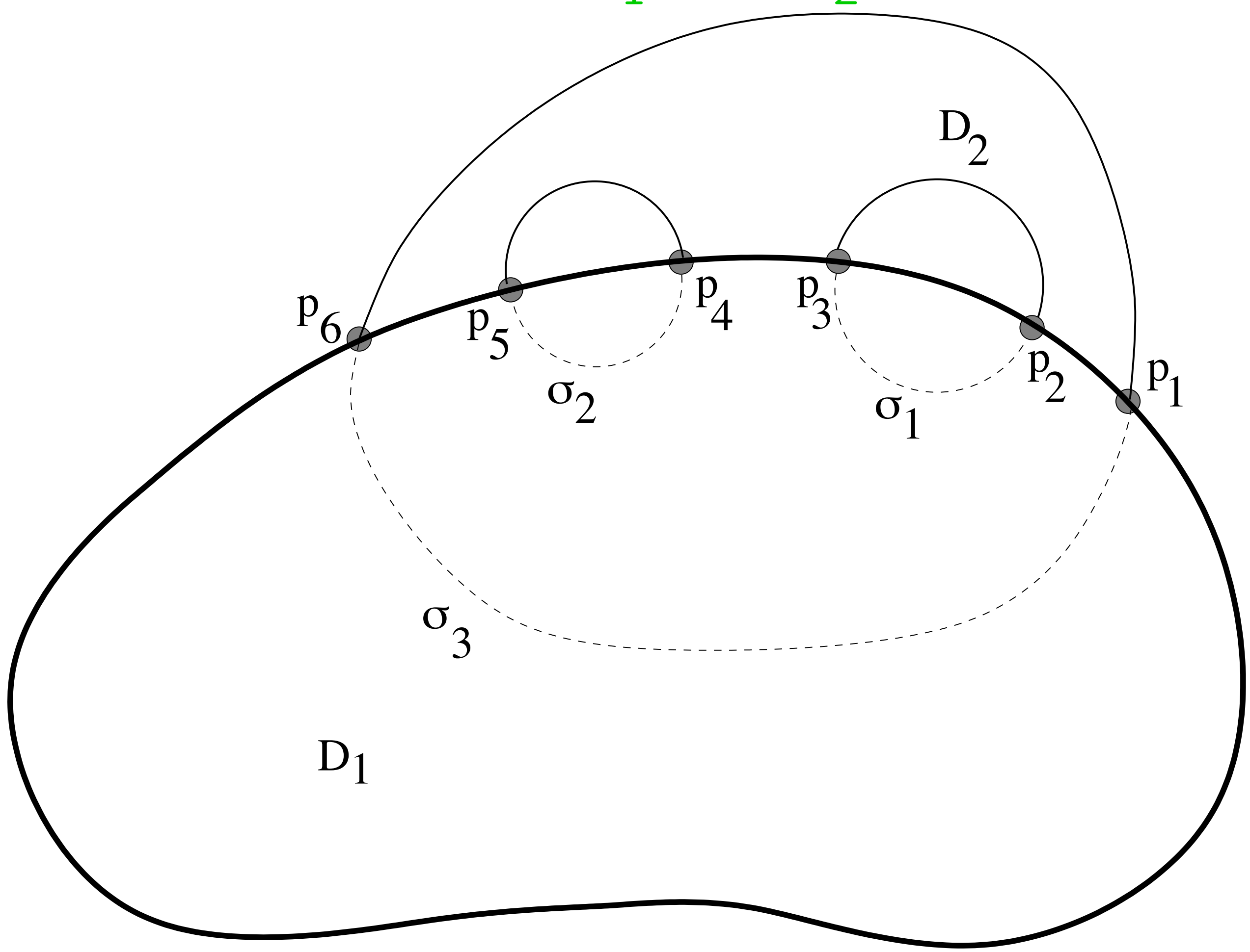
$$g(x) = c_1 \chi_{D_1}(x) + c_2 \chi_{D_2 \setminus D_1}(x), \qquad (c_1, c_2) \in \mathbb{R}^2$$

We have $g \in SBV_{\text{loc}}(\mathbb{R}^2)$ with $\nabla g \equiv 0$, and

$$J_g = \partial D_1 \bigcup (\partial D_2 \setminus D_1)\,.$$

We refer to J_g as the set of visible boundaries of the image g. The set $\partial D_2 \setminus D_1$ is the visible part of the boundary ∂D_2.

SETS D_1 and D_2



CASE M=1: ELASTICA

$\partial D_1 \cap \partial D_2 = \{p_1, p_2\}$: we consider the curves joining p_1 and p_2.

$\Sigma(\{D_1, D_2\})$ is the set of all curves σ of class $W^{2,p}$ such that

$\sigma(0) = p_1$, $\sigma(1) = p_2$;
$\frac{d\sigma}{dt}(0)$ is parallel to the tangent line $T_{p_1}(\partial D_2)$ of ∂D_2 at p_1,
$\frac{d\sigma}{dt}(1)$ is parallel to the tangent line $T_{p_2}(\partial D_2)$ of ∂D_2 at p_2.

The variational problem

$$(\mathcal{P})_1 \qquad \min\{\mathcal{F}(\sigma) : \sigma \in \Sigma(\{D_1, D_2\})\}$$

has a solution, which is called an elastica curve.

SPECIAL ASSUMPTION ON ELASTICA

Let $\widehat{\sigma}$ be a solution of the variational problem $(\mathcal{P})_1$ such that

$$(\widehat{\sigma}) \subset \overline{D}_1$$

If $\widehat{\sigma}$ is simple then the set $(\widehat{\sigma}) \cup (\partial_{\text{ext}} D_2 \setminus D_1)$ can be parameterized by means of a closed simple curve $\widehat{\gamma}$.

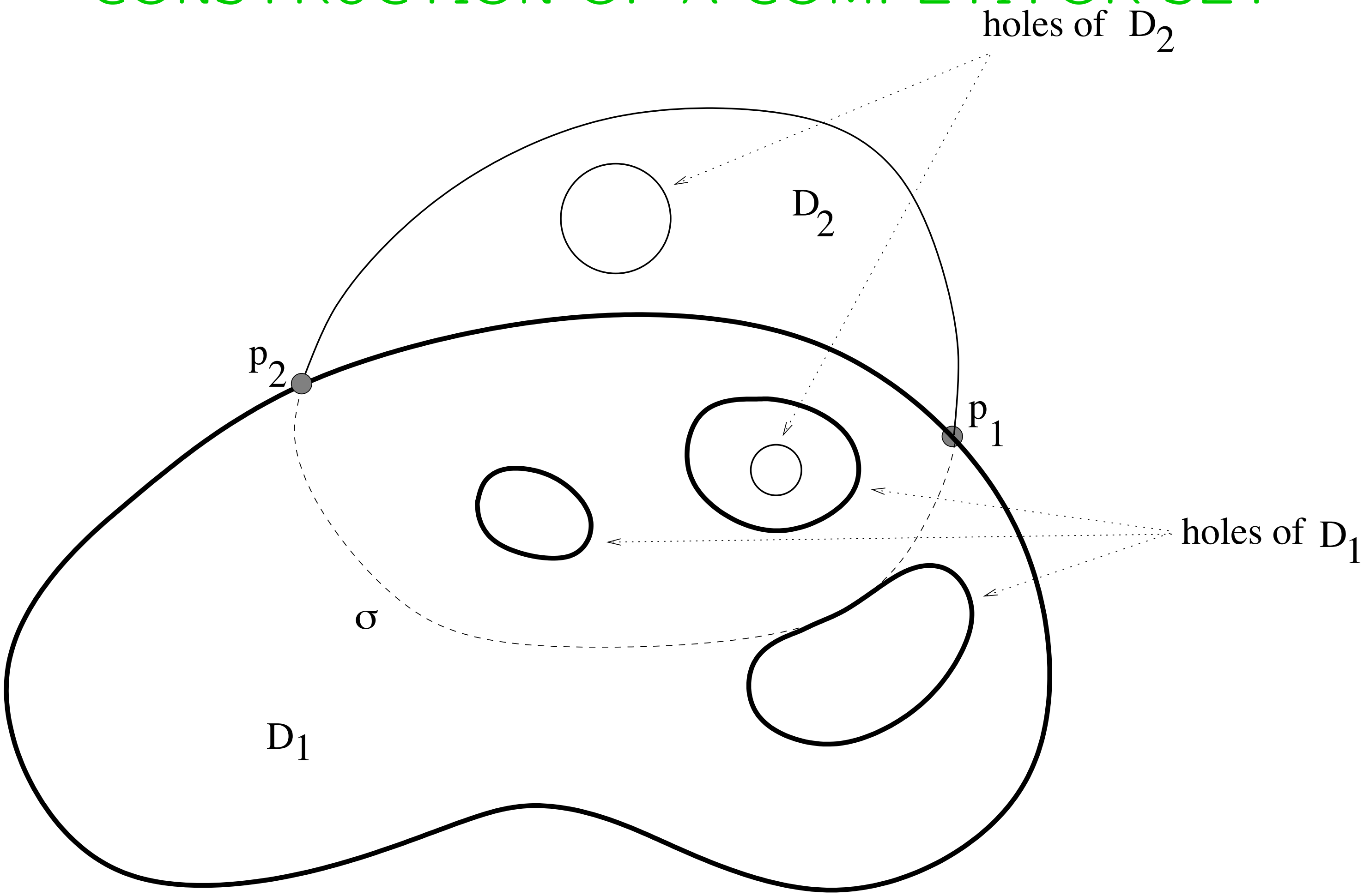
We say that the set of visible boundaries J_g admits a simple completion if there exists a simple curve $\widehat{\sigma}$ solving the variational problem $(\mathcal{P})_1$ and such that $(\widehat{\sigma}) \subset \overline{D}_1$, and

$$\partial_{\text{int}} D_2 \setminus D_1 \subset \{x \in \mathbb{R}^2 \text{ which are inside } \widehat{\gamma}\}.$$

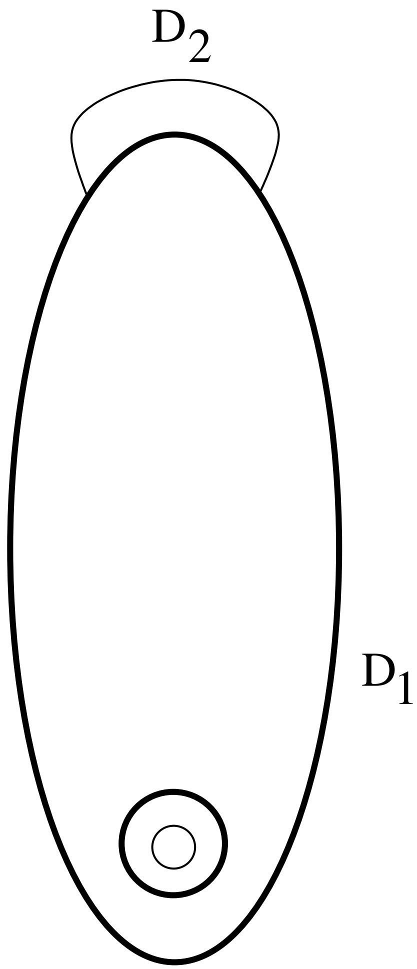
The inside of $\widehat{\gamma}^1$ is the set of points of index $I(\widehat{\gamma}, x) = 1$.

The visible holes of D_2 are in the inside of $\widehat{\gamma}$.

CONSTRUCTION OF A COMPETITOR SET



A SET THAT WE ARE NOT ABLE TO RECONSTRUCT (FAR HOLE)



PROBLEM

Starting from the grey level datum g we try to understand when the functional

$$\mathcal{G}_\lambda(u, n, \{E_1, \ldots, E_n\}, \psi) = \lambda \int_{\mathbb{R}^2} (u-g)^2 dx + \sum_{i=1}^{n} \overline{\mathcal{F}}(E_i)$$

is asymptotically (as $\lambda \to +\infty$) minimized by just two sets:

$$n = 2, \qquad E_1 = D_1, \quad E_2 = \widehat{D}_2$$

$\widehat{D}_2$ is constructed by completing the visible boundaries of D_2

$$\partial D_2 \setminus D_1$$

with an elastica connecting the points p_1, p_2.

Then ψ will give an information about the visible regions of sets.

RESULTS IN THE CASE $M = 1$

Let w be the collection

$$w := (u, n, \{E_1, \ldots, E_n\}, \psi).$$

We denote by $\mathcal{W}$ the set of all collections w. The domain of the functional $\mathcal{G}_\lambda$ is

$$\mathcal{D} := \left\{ w \in \mathcal{W} : J_u \subseteq J_\psi \subseteq \cup_{i=1}^n \partial^* E_i \right\},$$

The constraint $J_u \subseteq J_\psi$ is not closed.

Theorem 1 *Assume that the set of visible boundaries J_g admits a simple completion. Then we have*

$$\begin{aligned} \lim_{\lambda \to +\infty} \inf_{w \in \mathcal{D}} \mathcal{G}_\lambda(w) &= \int_{J_g \setminus \{p_1, p_2\}} [1 + |\kappa(x)|^p] \, d\mathcal{H}^1(x) \\ &+ \min_{\sigma \in \Sigma(\{D_1, D_2\})} \mathcal{F}(\sigma). \end{aligned}$$

RESULTS IN THE CASE $M = 1$

Proposition 1 *Assume that the set of visible boundaries J_g admits a simple completion. Then we have*

$$\min_{(n,\{E_1,\dots,E_n\})}\left\{\sum_{i=1}^{n}\overline{\mathcal{F}}(E_i)\,:\,J_g\subseteq\cup_{i=1}^{n}\partial^* E_i\right\}=\mathcal{F}(D_1)+\overline{\mathcal{F}}(\widehat{D}_2)$$

$$=\int_{J_g\setminus\{p_1,p_2\}}[1+|\kappa(x)|^p]\,d\mathcal{H}^1(x)+\min_{\sigma\in\Sigma(\{D_1,D_2\})}\mathcal{F}(\sigma).$$

RESULTS IN THE CASE $M = 1$

Collecting the previous results we obtain a corollary, which shows the link between the asymptotic property of functional $\mathcal{G}_\lambda$ and the variational problem considered in Proposition 1.

Corollary 1 *Assume that the set of visible boundaries J_g admits a simple completion. Then we have*

$$\begin{aligned} \lim_{\lambda \to +\infty} \inf_{w \in \mathcal{D}} \mathcal{G}_\lambda(w) &= \min_{(n,\{E_1,\dots,E_n\})} \left\{ \sum_{i=1}^{n} \overline{\mathcal{F}}(E_i) : J_g \subseteq \cup_{i=1}^{n} \partial^* E_i \right\} \\ &= \mathcal{F}(D_1) + \overline{\mathcal{F}}(\widehat{D_2}). \end{aligned}$$

CASE $M > 1$ AND D_1 WITH CONNECTED BOUNDARY

We assume that the set D_1 has connected boundary.

$$\partial D_1 \cap \partial D_2 = \{p_1, \dots, p_{2M}\}.$$

We denote T-junctions the points $p_1, \dots, p_{2M}$.

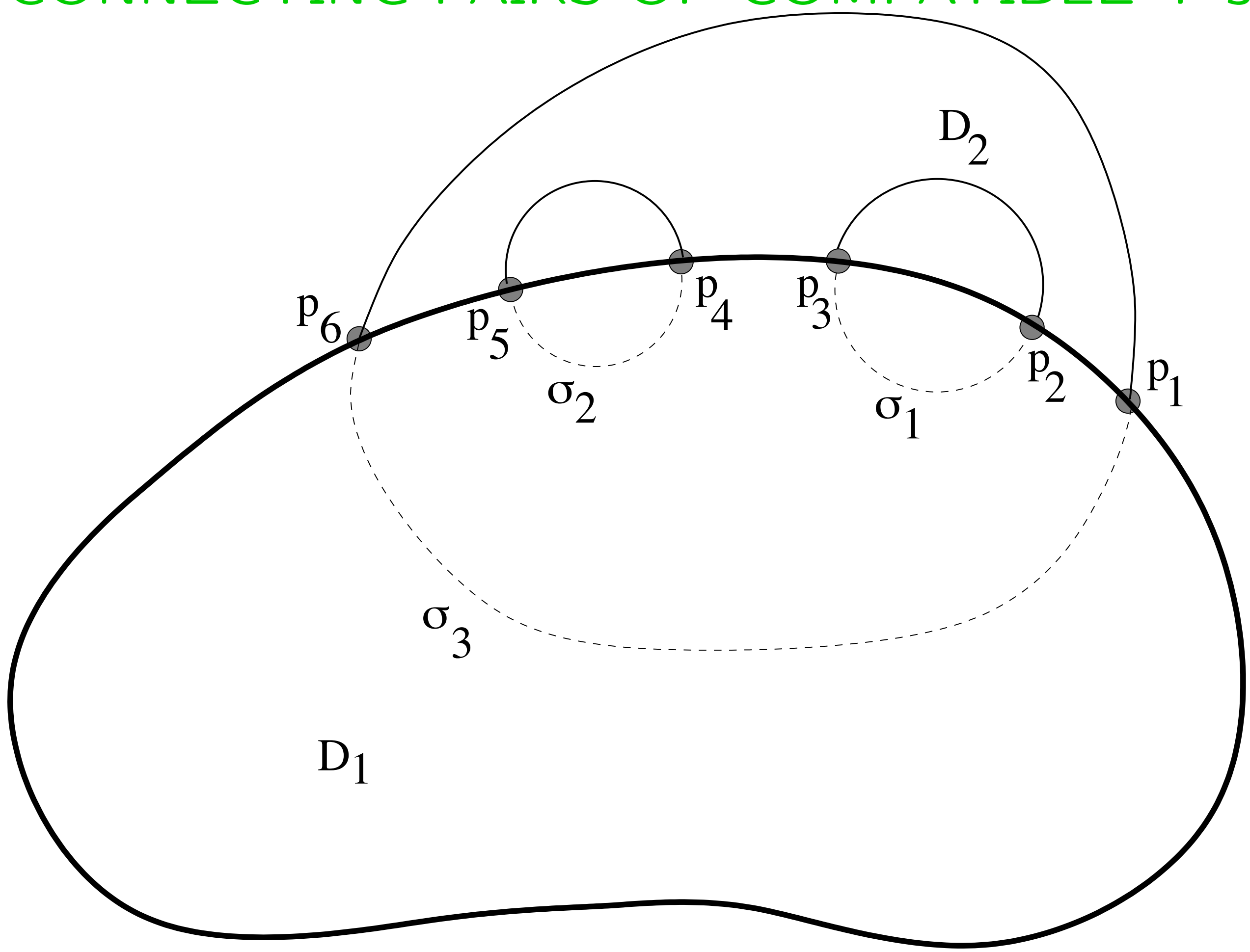
By means of an oriented parametrization of ∂D_1 the T-junctions $p_1, \dots, p_{2M}$ can be ordered along ∂D_1 in such a way that

$$p_1 < p_2 < \cdots < p_{2M}.$$

We say that two T-junctions p_i and p_j with $p_j > p_i$ are compatible if $j - i - 1$ is either 0 or an even integer.

Compatibility will permit us to consider families of elastica curves joining pairs of T-junctions without crossings.

ELASTICA CONNECTING PAIRS OF COMPATIBLE T-JUNCTIONS



ELASTICA CONNECTING PAIRS OF COMPATIBLE T-JUNCTIONS

$\Sigma(\{D_1, D_2\})$ is the set of families of curves $\{\sigma^1, \ldots, \sigma^M\}$ s.t.

$\sigma^i(0), \sigma^i(1) \in \{p_1, \ldots, p_{2M}\}$, with $\sigma^i(0)$ and $\sigma^i(1)$ compatible, $\forall i$;

there exists a bijective application
between $\{p_1, \ldots, p_{2M}\}$ and $\{\sigma^1(0), \sigma^1(1), \ldots, \sigma^M(0), \sigma^M(1)\}$

for any $i \in \{1, \ldots, M\}$
$\frac{d\sigma^i}{dt}(0)$ is parallel to the tangent line $T_{\sigma^i(0)}(\partial D_2)$ of ∂D_2 at $\sigma^i(0)$,
$\frac{d\sigma^i}{dt}(1)$ is parallel to the tangent line $T_{\sigma^i(1)}(\partial D_2)$ of ∂D_2 at $\sigma^i(1)$.

SPECIAL ASSUMPTION ON ELASTICAE

The variational problem

$$(\mathcal{P})_2 \qquad \min \left\{ \sum_{i=1}^{M} \mathcal{F}(\sigma^i) : \{\sigma^1, \ldots, \sigma^M\} \in \boldsymbol{\Sigma}(\{D_1, D_2\}) \right\}$$

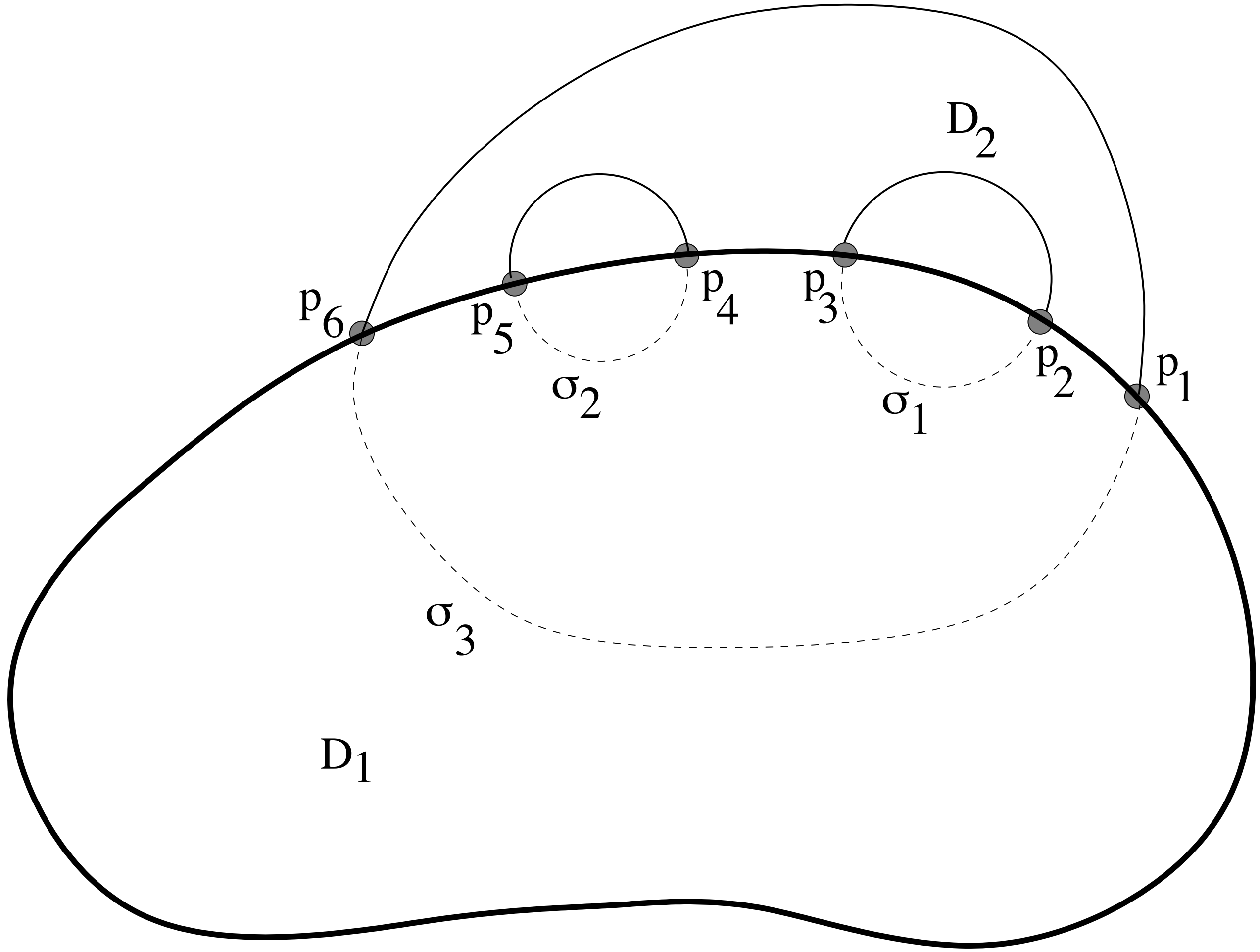
has a solution.

We say that the set of visible boundaries J_g admits a simple completion if there exists a family $\{\sigma^1, \ldots, \sigma^M\}$ of simple curves solving the variational problem $(\mathcal{P})_2$ such that $\sigma^i(0)$ and $\sigma^i(1)$ are compatible T-junctions for any i, and

$$\begin{aligned} (\sigma^i) \subset \overline{D}_1 \qquad & \text{for any } i \in \{1, \ldots, M\}, \\ \Big((\sigma^i) \cap (\sigma^j) \Big) \setminus \partial D_1 = \emptyset \qquad & \text{for any } i, j \in \{1, \ldots, M\}, \, i \neq j. \end{aligned}$$

The property that each elastica joins compatible T-junctions is a necessary condition in order that elasticae do not intersect.

CONSTRUCTION OF A COMPETITOR SET



RESULTS IN THE CASE $M > 1$

First, we prove an asymptotic result for the functional $\mathcal{G}_\lambda$ as $\lambda \to +\infty$.

Theorem 2 *Assume that the set of visible boundaries J_g admits a simple completion. Then we have*

$$\begin{aligned}\lim_{\lambda\to+\infty}\inf_{w\in\mathcal{D}}\mathcal{G}_\lambda(w) &= \int_{J_g\setminus\{p_1,\ldots,p_{2M}\}}[1+|\kappa(x)|^p]\,d\mathcal{H}^1(x)\\ &+ \min\left\{\sum_{i=1}^{M}\mathcal{F}(\sigma^i) : \{\sigma^1,\ldots,\sigma^M\}\in\Sigma(\{D_1,D_2\})\right\}.\end{aligned}$$

RESULTS IN THE CASE $M > 1$

Then we find a minimizer for the following variational problem.

Proposition 2 *Assume that the set of visible boundaries J_g admits a simple completion. Then we have*

$$\min_{(n,\{E_1,\dots,E_n\})} \left\{ \sum_{i=1}^{n} \overline{\mathcal{F}}(E_i) : J_g \subseteq \cup_{i=1}^{n} \partial^* E_i \right\} = \mathcal{F}(D_1) + \overline{\mathcal{F}}(\widehat{D_2})$$

$$= \int_{J_g \setminus \{p_1,\dots,p_{2M}\}} [1 + |\kappa(x)|^p] \, d\mathcal{H}^1(x)$$

$$+ \min \left\{ \sum_{i=1}^{M} \mathcal{F}(\sigma^i) : \{\sigma^1, \dots, \sigma^M\} \in \mathbf{\Sigma}(\{D_1, D_2\}) \right\}.$$

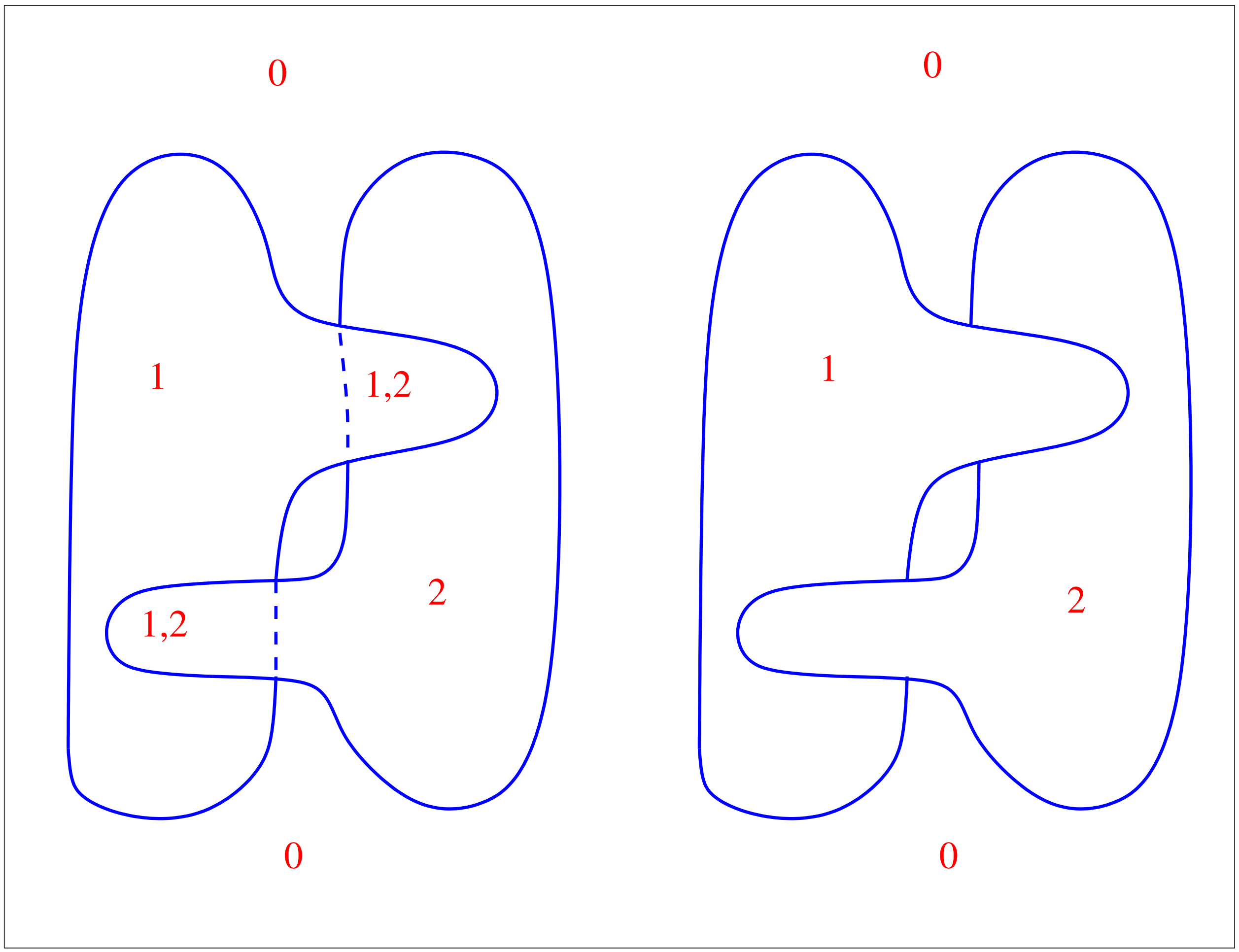
RESULTS IN THE CASE $M > 1$

Collecting the previous results we obtain the following corollary, which shows the link between the asymptotic property of functional $\mathcal{G}_\lambda$ and the variational problem considered in Proposition 2.

Corollary 2 *Assume that the set of visible boundaries J_g admits a simple completion. Then we have*

$$\begin{aligned}\lim_{\lambda \to +\infty} \inf_{w \in \mathcal{D}} \mathcal{G}_\lambda(w) &= \min_{(n,\{E_1,\ldots,E_n\})} \left\{ \sum_{i=1}^{n} \overline{\mathcal{F}}(E_i) : J_g \subseteq \cup_{i=1}^{n} \partial^* E_i \right\} \\ &= \mathcal{F}(D_1) + \overline{\mathcal{F}}(\widehat{D_2}).\end{aligned}$$

VALUES OF Ψ FOR INTERWOVEN SHAPES



INTERWOVEN SHAPES

Analogous results should be obtained for interwoven shapes, at least in the case of connected sets D_1 and D_2 having connected boundaries ∂D_1 and ∂D_2.

Work in progress...

APPROXIMATION BY Γ−CONVERGENCE

The building block of the variational model is the functional

$$\mathcal{F}(E) = \int_{\partial E} [1 + |\kappa|^2] d\mathcal{H}^1, \qquad (p = 2)$$

The functional $\mathcal{F}$ can be approximated by means of a family of functionals $(\mathcal{F}_\varepsilon)_\varepsilon$ in the sense of Γ-convergence:

$$\mathcal{F}_\varepsilon(u) = \int_\Omega \left[\varepsilon |\nabla u|^2 + \frac{V(u)}{\varepsilon}\right] dx + \frac{1}{2\varepsilon} \int_\Omega \left[2\varepsilon \Delta u - \frac{V'(u)}{\varepsilon}\right]^2 dx$$

where $\Omega \subset \mathbb{R}^2$ is a bounded image domain and the potential V is given by $V(u) = u^2(1-u)^2$.

When $\varepsilon \to 0^+$ the family of functionals $(\mathcal{F}_\varepsilon)_\varepsilon$ Γ-converges to the functional $\mathcal{F}$. The functionals $(\mathcal{F}_\varepsilon)_\varepsilon$ depend on a smooth function u and are more convenient for numerical computation.

RELAXATION OF THE GEOMETRIC PART

$$\mathcal{F}(E) = \int_{\partial E} [1 + |\kappa(x)|^p] \, d\mathcal{H}^1(x)$$

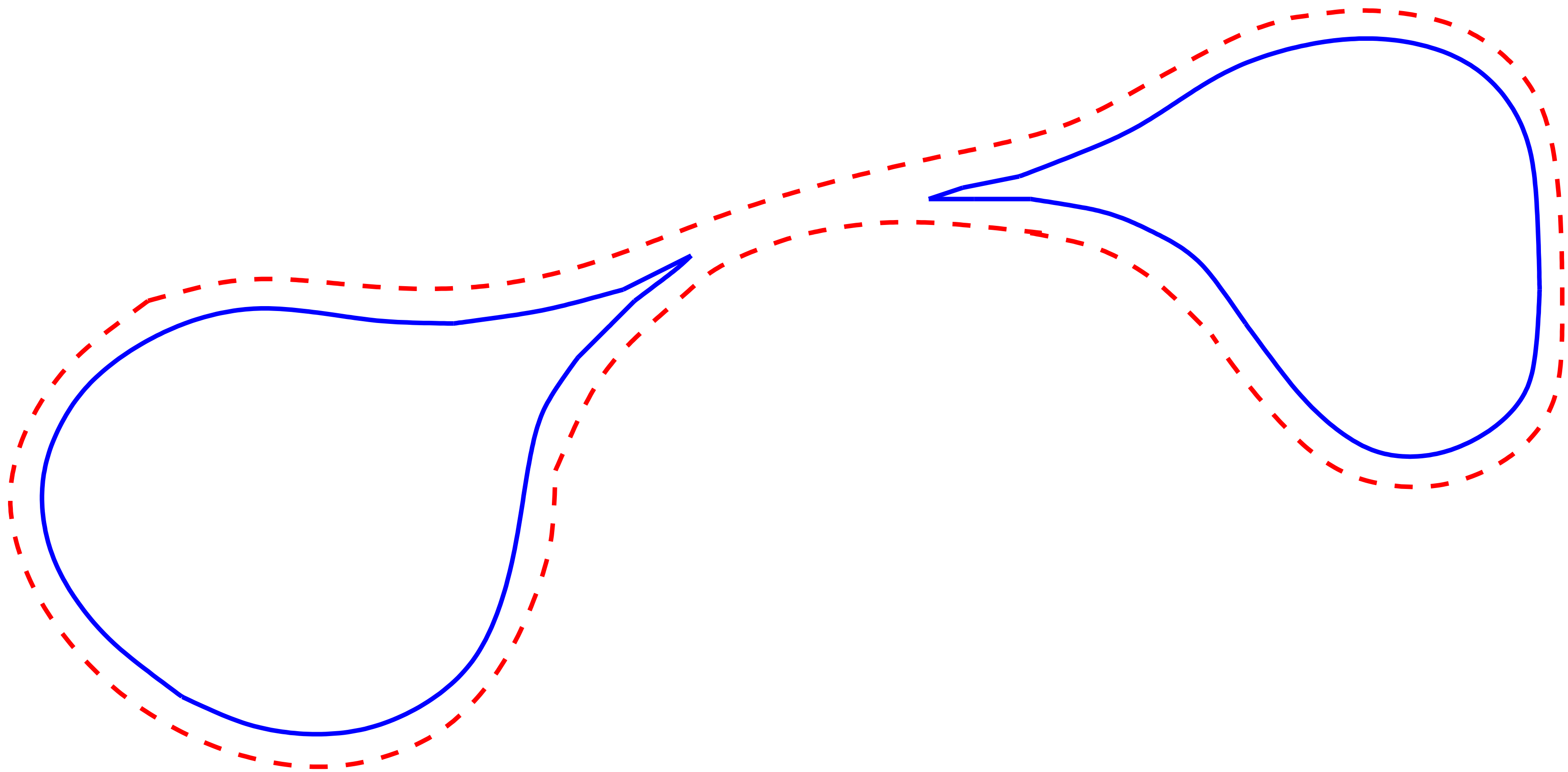
The functional $\mathcal{F}$ is not lower semicontinuous with respect to the $L^1(\mathbb{R}^2)$ convergence of characteristic functions of sets.

The functional $\mathcal{F}$ is defined on the family $\mathcal{M}$ of measurable sets

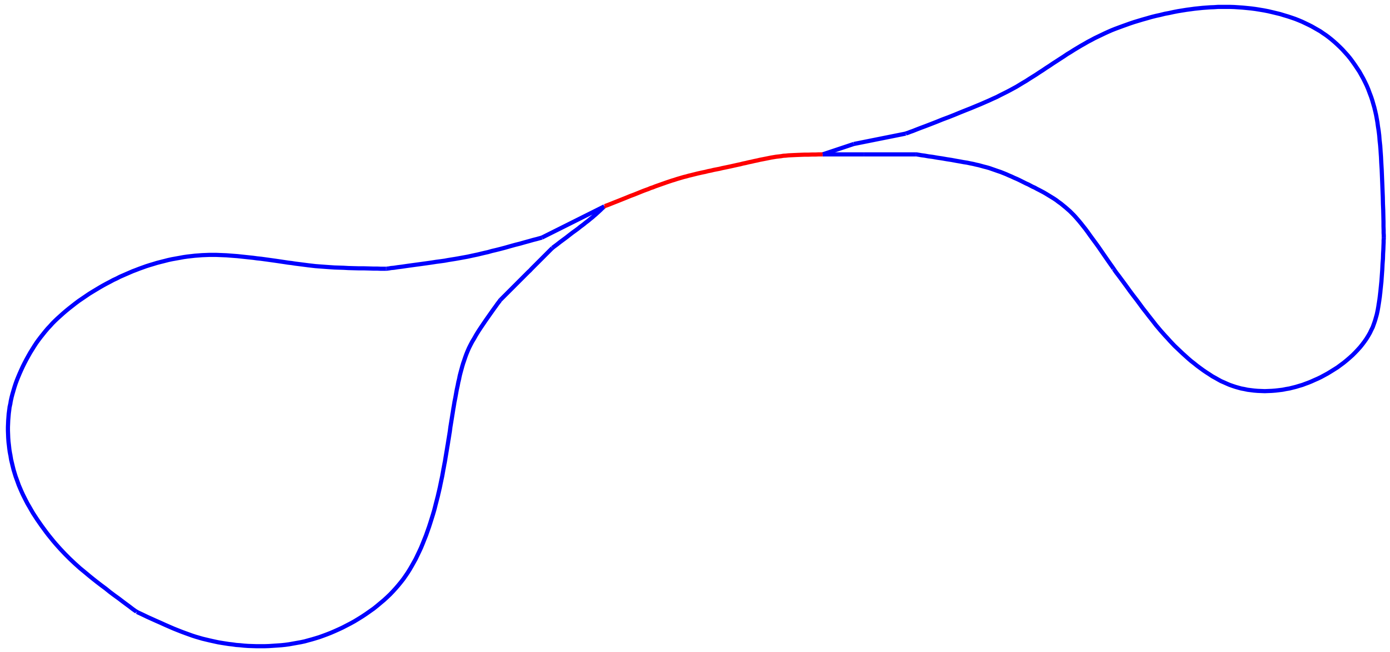
$$\mathcal{F}(E) := \begin{cases} \int_{\partial E} [1 + |\kappa(x)|^p] \, d\mathcal{H}^1(x) & \text{if } E \in \mathcal{C}^2(\mathbb{R}^2) \\ +\infty & \text{elsewhere on } \mathcal{M} \end{cases}$$

and it is relaxed (Bellettini, Dal Maso and Paolini (1993)):

$$\overline{\mathcal{F}}(E) = \inf \left\{ \liminf_{h \to +\infty} \mathcal{F}(E_h) : \{E_h\} \subset \mathcal{C}^2, \chi_{E_h} \to \chi_E \text{ in } L^1(\mathbb{R}^2) \right\}$$



THE CONSTRAINT $J_u \subseteq J_\psi$ IS NOT CLOSED



arc of multiplicity 2

THE FUNCTIONAL $\mathcal{F}$ ON SYSTEMS OF CURVES

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a closed curve of class $W^{2,p}$.

trace of γ: $\quad (\gamma) = \{\gamma(t) : t \in [0,1]\}$

A system Γ of curves is a finite family of closed curves of class $W^{2,p}$:

$$\Gamma = \{\gamma^1, \dots, \gamma^m\}, \qquad (\Gamma) = \bigcup_{i=1}^{m} (\gamma^i)$$

Functional $\mathcal{F}$ on the system Γ:

$$\mathcal{F}(\Gamma) = \sum_{i=1}^{m} \int_0^{l(\gamma^i)} [1 + |\kappa|^p] ds$$

where $l(\gamma)$ is the length of γ and s is the arclength.