Orbifolds in the Moduli Spaces of Extremal Kähler Metrics

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Orbifold compactness for extremal Kähler mertics

Theorem (X. Chen, B. Weber)
If (M_i, g_i) is a sequence of extremal Kähler manifolds with

Bounded energy: $\int |\,\mathrm{Rm}\,|^{\frac{n}{2}} < \Lambda$
Bounded diameters: $\mathrm{Diam}(M_i) < \delta$
Bounded volumes: $\mathrm{Vol}(M_i) > \nu$
Bounded Sobolev constants: $C_{M_i} < C_S$

Then a subsequence converges to a (reduced, C^∞) extremal Kähler orbifold.

Orbifolds and Riemannian Orbifolds

A topological orbifold is a (Hausdorff, second countable) topological space so that every point has a neighborhood homeomorphic to a subset of $\mathbb{R}^n/\Gamma$, where Γ is a finite group acting on $\mathbb{R}^n$.

A Riemannian orbifold (of differentiability class $C^{k,\alpha}$) is a differentiable orbifold so that a metric tensor exists at each manifold point, which can be completed on any orbifold cover to a metric of differentiability class $C^{k,\alpha}$.

If some other structure exists at every manifold point, the orbifold is said to carry that structure if it can be completed on the lift to any orbifold cover.

Role played by analysis

The sectional curvature restriction, required by the standard Riemannian convergence theorems, is very strong. Can we weaken it?

Main idea: consider manifolds on which Rm satisfies an elliptic system.

Then Analysis may lead to recovery of sectional curvature bounds, provided only much weaker assumptions.

Any Riemannian manifold satisfies,

$$\triangle \mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm} + \nabla^2 \mathrm{Ric}$$

But clearly this is not enough.

Naturality of elliptic systems

For example if $\nabla^2 \,\mathrm{Ric}$ disappears (as happens with Einstein metrics) we get

$$\triangle \,\mathrm{Rm} \;=\; \mathrm{Rm} * \mathrm{Rm}\,.$$

In other case we get more complicated elliptic systems, say for the Ricci tensor, which lets us control $\nabla^2 \,\mathrm{Ric}$.

Curvature tensors of various "Canonical metrics," minimizers of some energy functional

$$\int |\,\mathrm{Rm}\,|^2, \qquad \int |W|^2, \qquad \int R^2, \qquad \text{etc.},$$

tend to satisfy additional elliptic inequalities.

Form of the equations

Equations on an Extremal Kähler manifold:

$$\begin{aligned}\triangle\,\mathrm{Rm} &= \mathrm{Rm} * \mathrm{Rm} + \nabla^2\,\mathrm{Ric}\\ \triangle\,\mathrm{Ric} &= \mathrm{Rm} * \mathrm{Ric} + \nabla^2 R\\ \triangle\nabla R &= \mathrm{Ric} * \nabla R\end{aligned}$$

Previously studied Einstein case:

$$\triangle\,\mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}$$

Also CSC-Kähler, CSC-Bach Flat:

$$\begin{aligned}\triangle\,\mathrm{Rm} &= \mathrm{Rm} * \mathrm{Rm} + \nabla^2\,\mathrm{Ric}\\ \triangle\,\mathrm{Ric} &= \mathrm{Rm} * \mathrm{Ric}\,.\end{aligned}$$

The analytic technique is clearest in the Einstein case, where

$$\triangle u \geq -u^2$$

where $u = |\,\mathrm{Rm}\,|$.

Ordinarily one assumes the scale-invariant quantity $\int |\,\mathrm{Rm}\,|^{\frac{n}{2}}$ is bounded.

Consider

$$\triangle u \geq -fu.$$

If the Sobolev inequality is available, then

$$f \in L^{\frac{n}{2}} \Rightarrow u \in L^q_{loc}, \text{ all } q > 2$$
$$f \in L^p, \text{ some } p > \frac{n}{2} \Rightarrow u \in L^\infty$$

With $\triangle u \geq -u^2$, this theorem can't give us *pointwise* bounds on u.

Anderson (1989), BKN (1989), Tian (1990) managed to exploit the equation's *non*-linearity to partially recover the L^∞ bounds.

ϵ-Regularity

Theorem

There exists numbers ϵ_0, C so that

$$\int_{B(o,r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;\leq\; \epsilon_0$$

implies

$$\sup_{B_{r/2}} |\,\mathrm{Rm}\,| \;\leq\; C\, r^{-2} \left(\int_{B_r} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

The numbers ϵ and C depend on the Sobolev constant.

Proof: Modified version of Moser iteration.

C^0-orbifold compactness

Theorem

Assume M_i is a sequence of Riemannian manifolds that satisfy

* A sufficiently strong elliptic system for curvature
* Bounded energies: $\int_{M_i} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \leq \Lambda$
* Bounded diameters: $\mathrm{Diam}(M_i) \leq \delta$
* Bounded volumes: $\mathrm{Vol}(M_i) \geq \nu$
** A Sobolev constant bound: $C_{M_i} \leq C_S$
*** A local volume growth bound:

$$\mathrm{Vol}\, B(o,r) \leq v r^n \quad \text{when} \quad r < 1$$

Then a subsequence of M_i converges in the *GH*-topology to a manifold-with-singularities $\tilde{M}$.

The singularities are isolated, the number of singularities is bounded, and each is an orbifold point with a C^0 orbifold metric.

Sketch of Proof I: Weak Compactness

Let $bad_{r,i}$ be the set of points $p \in M_i$ so that $\int_{B(p,r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \geq \epsilon_0$.

Cover $bad_{r,i}$ by balls $B(p_{ij}, 2r)$ with $p \in \Omega_{r,i}$ and so the $B(p,r)$ are disjoint. There are at most Λ/ϵ_0 balls in any such covering.

Outside of these balls curvature is bounded by Cr^{-2}, so the manifolds $M_i - \left(\bigcup_j B(p_{ij}, 2r)\right)$ (sub)converge to a limit $\widetilde{M}_r$.

That $\widetilde{M}_r$ is nonempty is guaranteed by ***.

Define

$$\widetilde{M} = \overline{\bigcup_{r>0} \widetilde{M}_r}$$
$$S = \widetilde{M} - \bigcup_{r>0} \widetilde{M}_r.$$

Then $\widetilde{M}$ is a complete length space with singular set S.

Sketch of proof II: S consists of C^0 Riemannian orbifold points

Let $p \in S$, and let r denote distance from p.

Then $|\,\mathrm{Rm}\,| = o(r^{-2})$ near p, so any tangent cone at p is flat.

Also, p has a unique tangent cone; topologically it is a (one-point union of) standard cone(s) over $\mathbb{S}^{n-1}/\Gamma$.

The metric on the tangent cone is flat; if $n > 2$ it is therefore C^∞.

Thus the original metric is a C^0 orbifold metric (possibly nonreduced).

C^∞-Riemannian orbifold points, $n \geq 5$.

Lift to an orbifold cover B of $p \in S$.

We have seen that the metric can be completed on B to a C^0 metric.

We know $|\,\mathrm{Rm}\,| = o(r^{-2})$ near the singular point. If the regularity of Rm can be improved, so can the regularity of g.

Dimension 5 and higher: equations of the type $\triangle u \geq -fu$ yield removable singularity theorems; proof is entirely analytic.

Dimension $n = 4$ is the critical case. No purely analytic removable singularity theorem is possible.

C^∞-Riemannian orbifold points, $n = 4$.

The geometry itself must provide some help if we are to improve regularity in the $n = 4$ case.

One attempts to find an improved Kato inequality: for some $\alpha > 0$,

$$(1+\alpha)|\nabla|T||^2 \;\leq\; |\nabla T|^2.$$

From this one can derive an improved elliptic inequality,

$$|T|\triangle|T|^{1-\delta} \;\geq\; -(1-\delta)|T|^{1-\delta}|\triangle T|$$

when δ is small enough.

Metrics for which such an improved Kato inequality is possible include the Einstein, Yang-Mills, harmonic curvature, CSC-Kähler, and CSC-Bach flat metrics.
(ref: [Branson 2000], [Calderbank-Gouduchon-Herzlich 2000])

C^∞ orbifold regularity

In higher dimensions the removable singularity result goes through as before.

In dimension 4 we remove the singularity by using a new Kato inequality.

Letting $\overset{\circ}{\mathrm{Ric}}_{i\bar{j}} = \mathrm{Ric}_{i\bar{j}} - \frac{1}{m}\delta_{i\bar{j}}R$ denote the trace-free Ricci tensor, we find that

$$2\left|\nabla\left|\nabla \overset{\circ}{\mathrm{Ric}}\right|\right|^2 \leq \eta\left|\nabla^2 \overset{\circ}{\mathrm{Ric}}\right|^2 + \left|\overline{\nabla}\nabla \overset{\circ}{\mathrm{Ric}}\right|^2.$$

With knowledge about $X = \nabla R$ coming from the theory of extremal metrics, we boost $|\nabla \mathrm{Ric}|$ into a higher L^p space.

This is sufficient for Uhlenbeck's removable singularity result to go through.

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Application: Finding extremal Kähler metric on $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}^2}$

(Joint work with C. LeBrun and X. Chen)

Let $-H$ be the hyperplane class, and E_1, E_2 the classes of the exceptional divisors.

A typical symmetric Kähler class:

$$[\omega_{\alpha,\beta}] = s\,H + t\,(E_1 + E_2)$$

Put, say, $s = 1$ and

$$[\omega_t] \;=\; H + t\,(E_1 + E_2)\,.$$

When $t = 0$, one has the Fubini-Study metric on $\mathbb{CP}^2$. We want to show that given any t, $[\omega_t]$ possesses an Extremal representative.

By Arezzo-Pacard-Singer (2006), when t is sufficiently small, $[\omega_t]$ has an extremal representative

By LeBrun-Simanca (1994), the set of classes with extremal representatives is open.

The Chen-Weber result can be used to show closedness.

Justification of the Sobolev inequality is required.

One must show that curvature concentration does not occur.

We can recover the Sobolev inequality if the Yamabe constant is uniformly positive.

Idea of Tian's, modified by Chen-Weber:

$$Y^2 \ \geq \ C_1^2 - \frac{2}{3}\left(\frac{([\omega]\cdot C_1)^2}{[\omega\cdot[\omega]} + \|\mathcal{F}\|^2\right).$$

It is known that the Yamabe constant is positive. We have information on the Futaki invariant.

Thus we get a lower bound on Y.

This yields a Sobolev inequality of the form

$$\left(\int u^4\right)^{\frac{1}{2}} \ \leq \ \frac{6}{Y}\int |\nabla u|^2 + \frac{\max s}{Y}\int u^2$$

Lastly we show that the singularities cannot possibly form, essentially for energy reasons.

On Compact Kähler manifolds,

$$
\begin{aligned}
|\,\mathrm{Rm}\,|^2 &= \frac{5}{24}\int R^2 + 2\int |\,\overset{\circ}{\mathrm{Ric}}\,|^2 + \int |W^-|^2 \\
8\pi^2\chi &= \frac{1}{12}\int R^2 - \frac{1}{2}\int |\,\overset{\circ}{\mathrm{Ric}}\,|^2 + \int |W^-|^2 \\
12\pi^2\tau &= \frac{1}{24}\int R^2 - \int |W^-|^2.
\end{aligned}
$$

On ALE ZSC-Kähler manifolds,

$$
\begin{aligned}
|\,\mathrm{Rm}\,|^2 &= 2\int |\,\overset{\circ}{\mathrm{Ric}}\,|^2 + \int |W^-|^2 \\
8\pi^2\chi &= -\frac{1}{2}\int |\,\overset{\circ}{\mathrm{Ric}}\,|^2 + \int |W^-|^2 + \frac{8\pi^2}{|\Gamma|} \\
12\pi^2\tau &= -\int |W^-|^2 + \eta.
\end{aligned}
$$

Using $\int R^2 < 9 \cdot 32\pi^2$ and that $\tau = -1$ on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}$, we get

$$\int_{\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}} |W^-|^2 < 24\pi^2.$$

On any bubble, $b_1 = b_3 = 0$, and $b_2 = 1, 2$.

Only one bubble can possibly form, and it has $\mathbb{Z}_2$ symmetry.

Neither possibility for b_2 leads to a realizable bubble.