Noncompact Calabi-Yau Manifolds

Asymptotically Conical Ricci-flat Kähler Manifolds

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Introduction

- By *noncompact Calabi-Yau* I mean a noncompact Ricci-flat Kähler manifold whose end is asymptotic to a metric cone.
- A metric cone with base S is $C(S) = S \times \mathbb{R}_+$ with metric $g = dr^2 + r^2 g_S$.
- By definition $(C(S), g)$ is Ricci-flat Kähler $\Leftrightarrow$ (S, g_S) is Sasaki-Einstein.
- In other words, I am considering Ricci-flat Kähler manifolds with an end asymptotic to a cone over a Sasaki-Einstein manifold.
- These results are an extension of those of G. Tian and S.-T. Yau, '90, on the existence of Ricci-flat Kähler metrics on quasi-projective varieties $X \setminus D$, where $D \subset X$ with $\alpha[D] = -K_X, \alpha > 1$ admits a K-E metric.
  And also of results of D. Joyce, '99, on the existence of a Ricci-flat ALE metric on a crepant resolutions of $C^n/G, G \subset SL(n, \mathbb{C})$.

I will consider the following

Conjecture (J. Sparks)

Let $\pi : \hat{X} \to X$ be a crepant resolution of an isolated singularity $X = C(S)$, where $C(S)$ admits a Ricci-flat Kähler cone metric. Then $\hat{X}$ admits a unique Ricci-flat Kähler metric in each Kähler class in $H^2(\hat{X}, \mathbb{R})$ that is asymptotic to a cone over the Sasaki-Einstein manifold (S, g).

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Let $H^*_c(\hat{X}, \mathbb{R})$ denote the compactly supported cohomology.

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$$\nabla^k (\pi_* g - g_0) = O\left(r^{-2n+\delta-k}\right) \quad \text{on } \{x \in C(S) : r(x) > R\}, \tag{1}$$

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- The case of compact Kähler classes, $[\omega] \in H^2_c(\hat{X}, \mathbb{R})$, is precisely the case where there is "fast" convergence as in (1).
- If $\pi : \hat{X} \to X$ is a small resolution, i.e. $\operatorname{codim}_{\mathbb{C}}(E) > 1$, where $E = \pi^{-1}(o)$ is the exceptional set, then there are no Kähler classes in $H^2_c(\hat{X}, \mathbb{R})$.
  In particular, the conifold $X = \{z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}^4$ is the cone over $S^2 \times S^3$ with the homogeneous Sasaki-Einstein metric. Then X admits a crepant resolution $\pi : Y \to X$, where Y is the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1$. The exceptional set is $\mathbb{C}P^1 = \pi^{-1}(o)$. But Y admits a complete Ricci-flat Kähler metric converging to the cone with exponent $-2 - k$ in (1).

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Toric case

- There is a $(\mathbb{C}^*)^n$-action on $X = C(S)$ which restricts to an isometric T^n-action on S.
- $X = C(S)$ is Gorenstein and admits a toric Ricci-flat Kähler cone metric by A. Futaki, H. Ono, G. Wang.
- A crepant resolution $\pi : Y \to X$ is toric, and Y is described by a nonsingular simplicial fan $\tilde{\Delta}$ refining the convex polyhedral cone Δ defining X.
- A Kähler class in $H^2_c(Y, \mathbb{R})$ is given by a strictly convex support function $h \in \mathrm{SF}(\tilde{\Delta}, \mathbb{R})$ vanishing on the rays $u_j \in \mathbb{Z}^n$ defining Δ.

Corollary

Let $\pi : Y \to X$ be a crepant resolution of a Gorenstein toric Kähler cone X. Suppose the fan $\tilde{\Delta}$ defining Y admits a compact strictly convex support function. Then Y admits a unique Ricci-flat Kähler metric g, in this Kähler class, asymptotic to $(C(S), \bar{g})$ as in (1). Furthermore, g is invariant under T^n.

- The space of Ricci-flat Kähler metrics is $d = \dim H^2_c(Y, \mathbb{R})$ dimensional, where $d = \# P_\Delta \cap \mathbb{Z}^n$.
  $P_\Delta = \Delta \cap H_\gamma$, $H_\gamma = \{\gamma = -1\}$ is the hyperplane given by the Gorenstein condition.

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Motivation

Here are some possible motives for studying asymptotically conical Ricci-flat Kähler manifolds.

- *More general Ricci-flat manifolds*. If (M, g) is a complete Ricci-flat manifold with Euclidean volume growth, $\mathrm{Vol}(B_r(p)) \sim \Omega r^n$, then a pointed sequence $(M, p, r_j^{-2} g), r_j \to \infty$, has a subsequence that Gromov-Hausdorff converges to a metric cone M_∞.
- *Local mirror symmetry*. Many of the examples, such as toric cones $C(S)$, admit special Lagrangian fibrations. (M. Gross and E. Goldstein)
- *AdS/CFT*. This is a conjectured duality between type IIB string theory on $\mathrm{AdS}^5 \times S$ and a superconformal field theory on $\mathbb{R}^{1,3} \times C(S)$, where S is Sasaki-Einstein. The SCFTs are described by certain *quiver gauge theores*. These gauge theories describe the worldvolume theory for D3-branes placed at the cone singularity. This is they dependent on the algebraic geometry of the singularity and its resolutions.

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$\dim_{\mathbb{C}} X = 2$

The dimension 2 case is well known.
A Ricci-flat Kähler cone is $X = C(S) \cup \{o\} = \mathbb{C}^2/G, \quad G \subset U(2)$ with the quotient of the flat metric.
The singularity $o \in X$ is Gorenstein $\Leftrightarrow G \subset SU(2)$.
And X admits a crepant resolution $\pi : \hat{X} \to X \Leftrightarrow G \subset SU(2)$.
A fortiori $G \subset SU(2) \Leftrightarrow o \in X$ is a *canonical singularity*, i.e. there is a resolution $\pi : Y \to X$ with $K_Y = \pi^* K_X + \sum_i a_i E_i$ with $a_i \geq 0$.

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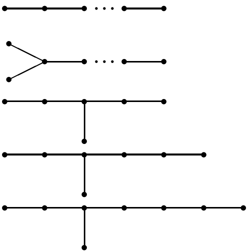
The classification is as follows.



	G	$\|G\|$	$f(x,y,z)$
A_k $k \geq 1$	cyclic	$k+1$	$x^{k+1}+y^2+z^2$
D_k $k \geq 4$	binary dihedral	$4(k-2)$	$x^{k-1}+xy^2+z^2$
E_6	binary tetrahedral	24	$x^3+y^4+z^2$
E_7	binary octahedral	48	$x^3+xy^3+z^2$
E_8	binary icosahedral	120	$x^3+y^5+z^2$

It is well known that each singularity $X = \mathbb{C}^2/G, \quad G \subset SL(2,\mathbb{C})$, admits a unique crepant resolution $\pi : \hat{X} \to X$.

$\mathbb{C}$-dimension 2

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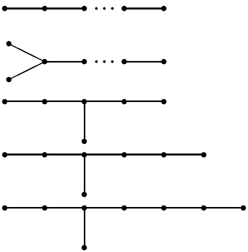

	G	$\lvert G \rvert$	$f(x,y,z)$
A_k $k \geq 1$	cyclic	$k+1$	$x^{k+1} + y^2 + z^2$
D_k $k \geq 4$	binary dihedral	$4(k-2)$	$x^{k-1} + xy^2 + z^2$
E_6	binary tetrahedral	24	$x^3 + y^4 + z^2$
E_7	binary octahedral	48	$x^3 + xy^3 + z^2$
E_8	binary icosahedral	120	$x^3 + y^5 + z^2$

It is well known that each singularity $X = \mathbb{C}^2/G, \quad G \subset SL(2,\mathbb{C})$, admits a unique crepant resolution $\pi : \hat{X} \to X$.

Ricci-flat ALE spaces

The resolution $\hat{X}$ admits a Ricci-flat Kähler metric, in each Kähler class, asymptotic to the flat metric g_0 on $\mathbb{C}^2/G$, i.e. an Asymptotically Locally Euclidean metric (ALE). This means

$$\nabla^k(\pi_* g - g_0) = O(r^{-2n-k}), \quad \text{where } r \text{ is the radius on } X = C^n/G.$$

Here $n = 2$, $SU(2) = Sp(1)$ so g is hyper-kähler, i.e. there are parallel complex structures J_1, J_2, J_3 with Kähler forms $\omega_1, \omega_2, \omega_3$.
So unlike the $n \geq 3$ case considered here, these metrics can be given explicitly by hyper-kähler reduction.

- The case A_1, $\hat{X} = T^*\mathbb{C}P^1$, is due to T. Eguchi and A. Hanson, '78.
- $A_k, k \geq 1$, gravitational multi-instantons, is due to G. Gibbons and S. Hawking, '78.
- $A_k, k \geq 1$, case was also proved by N. Hitchin, '79, using twistor methods.
- The general $n = 2$ case was proved by P. Kronheimer, '89, using hyper-Kähler quotients.
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Ricci-flat Kähler cones

Definition

A Riemannian manifold (S,g) of dimension $2n-1$ is Sasakian if the metric cone $(C(S),\bar{g})$, $C(S)=\mathbb{R}_{>0}\times S$ and $\bar{g}=dr^2+r^2g$, is Kähler.

- $\xi = J(r\frac{\partial}{\partial r})$ is Killing, and $\xi - iJ\xi = \xi + ir\frac{\partial}{\partial r}$ is a holomorphic vector field on $C(S)$. Either ξ
  - generates a free $U(1)$-action on S and S is *regular*,
  - generates a locally free $U(1)$-action on S and S is *quasi-regular*, or
  - the orbits do not close and S is *irregular*.
- $\eta = g(\xi,-) = 2d^c\log r$ is a contact form on S with Reeb vector field ξ.
- The foliation generated by ξ has a transverse Kähler structure with form $\omega^T=\frac{1}{2}d\eta$.
- The Kähler form of $\bar{g}$ is $\omega=\frac{1}{2}dd^cr^2$.

Proposition

Let (S,g) be a $2n-1$-dimensional Sasaki manifold. Then the following are equivalent.

(i) (S,g) is Sasaki-Einstein with the Einstein constant being necessarily $2n-2$.
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Ricci-flat Kähler cones

Given a smooth basic function $\phi \in C^\infty_B(S)$, we consider the following deformed Sasaki-structure.

$$\tilde{\eta} = \eta + 2d^c_B\phi, \quad om\tilde{e}ga^T = \omega^T + dd^c_B\phi, \quad \tilde{\xi} = \xi. \tag{2}$$

Let $\tilde{r} = r \exp \phi$. Then $\tilde{\omega} = \frac{1}{2}dd^c\tilde{r}^2$ is the new Kähler form on $C(S)$.

Proposition

The following necessary conditions for S to admit a deformation of the transverse Kälher structure to a Sasaki-Einstein metric are equivalent.

- (i) $c_1^B = a[d\eta]$ *for some positive constant a.*
- (ii) $c_1^B > 0$, *i.e. represented by a positive* $(1,1)$*-form, and* $c_1(D) = 0$.
- (iii) *For some positive integer* $\ell > 0$, *the* ℓ*-th power of the canonical line bundle* $\mathbf{K}^\ell_{C(S)}$ *admits a nowhere vanishing section* Ω *with* $\mathcal{L}_\xi\Omega = in\Omega$.

Then $X = C(S) \cup \{o\}$ is ℓ-Gorenstein if Proposition 2.3 is satisfied, Gorenstein if $\ell = 1$.

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Toric Sasaki-Einstein manifolds

Definition

A Sasaki manifold (S, g) of dimension $2n - 1$ is toric if there is an effective action of an n-dimensional torus $T = T^n$ preserving the Sasaki structure such that the Reeb vector field ξ is an element of the Lie algebra $\mathfrak{t}$ of T.
Equivalently, a toric Sasaki manifold is a Sasaki manifold S whose Kähler cone $C(S)$ is a toric Kähler manifold.

We have an effective holomorphic action of $T_{\mathbb{C}} \cong (\mathbb{C}^*)^n$ on $C(S)$ whose restriction to $T \subset T_{\mathbb{C}}$ preserves the Kähler form $\omega = d(\frac{1}{2}r^2\eta)$. So there is a moment map

$$\begin{aligned} \mu : C(S) &\longrightarrow \mathfrak{t}^* \\ \langle \mu(x), X \rangle &= \frac{1}{2}r^2\eta(X_S(x)), \end{aligned} \tag{3}$$

We have the moment cone defined by

$$\mathcal{C}(\mu) := \mu(C(S)) \cup \{0\}, \tag{4}$$

There are vectors $u_i, i = 1, \ldots, d$ in the integral lattice $\mathbb{Z}_T = \ker\{\exp(2\pi i \cdot) : \mathfrak{t} \to T\}$ such that

$$\mathcal{C}(\mu) = \bigcap_{j=1}^{d} \{y \in \mathfrak{t}^* : \langle u_j, y \rangle \geq 0\}. \tag{5}$$

Toric Sasaki-Einstein manifolds

Each face $\mathcal{F} \subset \mathcal{C}(\mu)$ is the intersection of a number of facets $\{y \in \mathfrak{t}^* : l_{J_k}(y) = \langle u_{j_k}, y\rangle = 0\}, k = 1, \dots, e$.
Then S is smooth, and the cone $\mathcal{C}(\mu)$ is said to be *non-singular* if and only if

$$\left\{ \sum_{k=1}^{a} \nu_k u_{j_k} : \nu_k \in \mathbb{R} \right\} \cap \mathbb{Z}_T = \left\{ \sum_{k=1}^{a} \nu_k u_{j_k} : \nu_k \in \mathbb{Z} \right\} \tag{6}$$

for all faces $\mathcal{F}$. The dual cone to $\mathcal{C}(\mu)$ is

$$\mathcal{C}(\mu)^* = \{\tilde{x} \in \mathfrak{t} : \langle \tilde{x}, y\rangle \geq 0 \text{ for all } y \in \mathcal{C}(\mu)\}. \tag{7}$$

$\mathcal{C}(\mu)^*$ is spanned by $u_i, i = 1, \dots, d$ and is the cone defining $(S) \cup \{o\}$ as an affine toric variety.

Proposition (J. Sparks, S.-T. Yau)

Let S be a compact toric Sasaki manifold and $C(S)$ its Kähler cone. For any $\xi \in \operatorname{Int} \mathcal{C}(\mu)^*$ there exists a toric Kähler cone metric, and associated Sasaki structure on S, with Reeb vector field ξ. And any other such structure is a transverse Kähler deformation, i.e. $\tilde{\eta} = \eta + 2d^c\phi$, for a basic function ϕ.

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Toric Sasaki-Einstein manifolds

The topological necessary condition for a Sasaki-Einstein metric is the following.

Proposition

Let S be a compact toric Sasaki manifold of dimension $2n-1$. Then the conditions of Proposition 2.3 are equivalent to the existence of $\gamma \in \mathfrak{t}^*$ such that

- *(i)* $(\gamma, u_k) = -1$, *for* $k = 1, \dots, d$,
- *(ii)* $(\gamma, \xi) = -n$, *and*
- *(iii)* *there exists* $\ell \in \mathbb{Z}_+$ *such that* $\ell\gamma \in \mathbb{Z}_T^* \cong \mathbb{Z}^n$

Then there is nowhere vanishing section of $\mathbf{K}^{\ell}_{C(S)}$. *And* $C(S)$ *is* ℓ*-Gorenstein if and only if a* γ *satisfying the above exists.*

Toric Sasaki-Einstein manifolds

Theorem (A. Futaki, H. Ono, G. Wang)

Suppose S is a toric Sasaki manifold satisfying Proposition 2.6. Then we can deform the Sasaki structure by varying the Reeb vector field and then performing a transverse Kähler deformation to a Sasaki-Einstein metric. The Reeb vector field and transverse Kähler deformation are unique up to isomorphism.

One varies the Reeb vector ξ in $\{x \in \mathfrak{t} : \gamma(x) = -n\} \cap \mathcal{C}(\mu)^*$ to minimize the volume of

$$\Delta_\xi := \{y \in \mathfrak{t}^* : (y, \xi) = \frac{1}{2}\} \cap \mathcal{C}(\mu). \tag{8}$$

The volume of the Sasaki manifold S_ξ with Reeb vector field ξ is

$$\mathrm{Vol}(S_\xi) = 2n(2\pi)^n \,\mathrm{Vol}(\Delta_\xi). \tag{9}$$

This cancels out the Futaki invariant of the tranversal Kähler structure, which is the only obstruction to a Sasaki-Einstein metric according to Theorem 2.7.

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Resolutions

We consider Kähler cones $X = C(S) \cup \{o\}$ which are Gorenstein. So that the dualizing sheaf $\omega_X \cong i_*(\mathcal{O}(\mathbf{K}_{C(S)}))$, where $i : C(S) \to X$ is the inclusion, is trivial. And we consider resolutions $\pi : \hat{X} \to X$ which are *crepant*

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From the following result of H. Laufer and D. Burns the singularity of X is *rational*.

Proposition

Let Ω be a holomorphic n-form defined, and nowhere vanishing, on a deleted neighborhood of $o \in X$. Then $o \in X$ is rational if and only if

$$\int_U \Omega \wedge \bar{\Omega} < \infty, \tag{11}$$

for U a sufficiently small neighborhood of $o \in X$.

Recall this means $R^i\pi_*\mathcal{O}_Y = 0$, for $i > 0$, for any resolution $\pi : Y \to X$.

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Some properties of $\pi : \hat{X} \to X$

We collect some properties of a *crepant* resolution $\pi : \hat{X} \to X$.

- Since $o \in X$ is rational, the Leray spectral sequence shows $H^i(\hat{X}, \mathcal{O}) = 0$, for $i > 0$.
- Thus $\operatorname{Pic} \hat{X} = H^2(\hat{X}, \mathbb{Z})$, by the exponential cohomology sequence.
- The *divisor class group* $\operatorname{Cl}(X, o) := \varinjlim \frac{\operatorname{WDiv} U}{\operatorname{CDiv} U}$, $U \ni o$. By H. Flenner $\operatorname{Cl}(X, o) = H^2(S, \mathbb{Z})$.
- We define $\rho(X) := \operatorname{rank} \operatorname{Cl}(X, o)$. Thus $\rho(X) = b_2(S)$.
- The number of π-exeptional divisors, denoted $c(X)$, is independent of the crepant resolution $\pi : \hat{X} \to X$.

Proposition

Let $X = C(S) \cup \{o\}$ have a crepant resolution $\pi : \hat{X} \to X$, then

(i) $b_1(\hat{X}) = b_{2n-1}(\hat{X}) = b_{2n}(\hat{X}) = 0$,
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When $\dim X = 3$ two crepant resolutions differ by a finite sequence of birational modifications called flops.
These preserve $H^*(\hat{X}, \mathbb{Z})$, so it is invariant of the resolution.

Some properties of $\pi : \hat{X} \to X$

We collect some properties of a *crepant* resolution $\pi : \hat{X} \to X$.

- Since $o \in X$ is rational, the Leray spectral sequence shows $H^i(\hat{X}, \mathcal{O}) = 0$, for $i > 0$.
- Thus $\mathrm{Pic}\,\hat{X} = H^2(\hat{X}, \mathbb{Z})$, by the exponential cohomology sequence.
- The *divisor class group* $\mathrm{Cl}(X, o) := \varinjlim \frac{\mathrm{WDiv}\,U}{\mathrm{CDiv}\,U}$, $U \ni o$. By H. Flenner $\mathrm{Cl}(X, o) = H^2(S, \mathbb{Z})$.
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Approximate metric

Let $\pi : Y \to X$ be a resolution of $o \in X = C(S) \cup \{o\}$. Let $\overline{Y}_a = \{y \in Y : r(y) \leq a\} \subset Y$, $a > 0$ with $\partial \overline{Y}_a = S_a$. Then

$$H_c^*(Y, \mathbb{R}) = H^*(\overline{Y}_a, S_a, \mathbb{R}), \quad \text{and } H_c^2(Y, \mathbb{R}) \subset H^2(Y, \mathbb{R}). \tag{12}$$

Proposition

Let $\pi : Y \to X$ be a resolution of the Kähler cone $X = C(S)$. Let g be a Kähler metric on Y with Kähler form ω. Suppose

$$\|\pi_* g - \bar{g}\|_{\bar{g}} = O\left(r^{-\alpha}\right), \tag{13}$$

where $\bar{g}$ is the cone metric on $C(S)$. If $\alpha > 2$, then $[\omega] \in H_c^2(Y, \mathbb{R})$.

Let $\bar{\omega} = \frac{1}{2} dd^c r^2$, and set $\beta = \omega - \bar{\omega}$.
Let $C \in H_2(S, \mathbb{R})$, then

$$\int_C \beta = \int_C |i^* \beta|_{\bar{g}|_C} \mu_{\bar{g}|_C} \leq C \int_C r^{-\alpha+2} \mu_{g_S} \to 0 \tag{14}$$

as $r \to \infty$.

Approximate metric

Lemma

Let $\tilde{\omega}$ be a Kähler metric on Y whose cohomology class $[\tilde{\omega}] \in H^2_c(Y, \mathbb{R})$. Then there exists a Kähler metric ω_0 on Y with $[\omega_0] = [\tilde{\omega}]$ and for some $r_0 > 0$ on $Y_{r_0} = \{y \in Y : r(y) \geq r_0\}$ ω_0 restricts to $\pi^*\omega$, the pull-back of the Ricci-flat Kähler metric.

$[\tilde{\omega}]$ is Poincarè dual to $\sum_i a_i D_i$ for $a_i \in \mathbb{R}$ where $\{D_i\}$ are the prime divisors in $E = \pi^{-1}(o)$. There exists a closed $(1,1)$-form β $[\beta] = [\tilde{\omega}]$. Because $o \in X$ is a rational singularity and X is Stein, the Leray spectral sequence implies that $H^j(Y, \mathcal{O}_Y) = 0$ for $j > 0$. So there is a $u \in C^\infty(Y)$ with $\mathrm{i}\partial\bar{\partial}u = \tilde{\omega} - \beta$. Then for a cut-off function $\phi : Y \to [0,1]$ and a convex function $\nu : Y \to \mathbb{R}$

$$\omega_0 = \beta + i\partial\bar{\partial}(\phi u) + Ci\partial\bar{\partial}(\nu(\frac{r^2}{2})), \text{ for } C > 0. \qquad (15)$$

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Monge-Ampère equation

Let $\pi : Y \to X$ be a crepant resolution of a Ricci-flat Kähler cone $X = C(S) \cup \{o\}$.
There is a holomorphic n-form Ω on X satisfying

$$c\Omega \wedge \bar{\Omega} = \omega^n. \tag{16}$$

Let Ω also denote the extension of $\pi^*\Omega$ to a nowhere vanishing n-form on Y.
Define a real valued function

$$f = \log\left(\frac{c\Omega \wedge \bar{\Omega}}{\omega_0^n}\right), \tag{17}$$

Then $f = 0$ outside the compact set $\bar{Y}_{r_0} = \{y \in Y : r(y) \leq r_0\}$, and $i\partial\bar{\partial}f = \mathrm{Ricci}(\omega_0)$.
A Ricci-flat Kähler metric is equivalent to a solution to the Monge-Ampère equation:

$$\begin{cases} (\omega_0 + i\partial\bar{\partial}\phi)^n = e^f\omega_0^n, \\ \omega_0 + i\partial\bar{\partial}\phi > 0. \end{cases} \tag{18}$$

Existence

The following is due to G. Tian and S.-T. Yau.

Proposition

Let ω_0 be the Kähler form defined in Lemma 2.11. Then there is a unique solution ϕ to (18) such that $\phi(y)$ converges uniformly to zero as y goes to infinity, and there is a constant $c > 1$ so that $c^{-1}\omega_0 < \omega_0 + i\partial\bar{\partial}\phi < c\omega_0$. It follows that $\tilde{\omega} = \omega_0 + i\partial\bar{\partial}\phi$ is a complete Ricci-flat Kähler metric on Y.

Asymptotics of the metric

Lemma

Let ϕ be the solution to (18) given in Proposition 3.1. For any $\delta > 0$ there are constants $C, C_\delta > 0$ so that

$$-C_\delta(1+r^2(y))^{-n+1}(\log r(y))^\delta \leq \phi(y) \leq C(1+r^2(y))^{-n+1}, \quad \text{for } y \in Y_{r_0}, \tag{19}$$

where Y_{r_0} is as in Lemma 2.11.

Set $\rho = Kr^{-2n+2}$. Then computation shows that

$$(\omega_0 + dd^c\rho)^n \leq \omega_0^n, \quad \text{for } K > 0 \text{ and } 2K(n-1) \leq r^{2n} \tag{20}$$

The maximum principle give the upper bound in (19).
For the lower set $\rho = Kr^{-2n+2}(\log r)^\delta$, and

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for $K < 0$ and r sufficiently large. An application of the maximum principle gives the lower bound.

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Proposition

Let ϕ be as above. Then for $\frac{1}{2} > \delta > 0$, there are constants $C_{\delta,k}$ depending only on k and δ so that

$$\|\nabla^k \phi\|_{g_0}(y) \leq C_{\delta,k} r(y)^{-2n+2-k+\delta}, \quad \text{for } , y \in Y_{r_0}. \tag{22}$$

Consider the elliptic operator P defined by

$$(Pu)\omega_0^n := i\partial\bar{\partial}u \wedge (\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}). \tag{23}$$

On Y_{r_0}, $g_0 = dr^2 + r^2 g$, the cone metric, and the Euler vector field $r\partial_r$ generates an action of $\mathbb{R}_{>0}$ by homothetic isometries on g_0. For $a > 1$ denote this action by $\psi_a : Y_{r_0} \to Y_{r_0}$.

$$\psi_a^* g_0 = a^2 g_0. \tag{24}$$

Then we have a weighted version of the schauder estimates for $P(u) = w$

$$\|u\|_{C^{k+2,\alpha}_{\beta}} \leq C \left(\|w\|_{C^{k,\alpha}_{\beta-2}} + \|u\|_{C^0_{\beta}} \right), \tag{25}$$

where $C^{k,\alpha}_{\beta}$ is the *weighted* Hölder space. Apply (25) to $P(\phi) = e^f - 1$ with $\beta = -2n + 2 + \delta$.

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Proposition

Let ϕ be as above. Then for $\frac{1}{2} > \delta > 0$, there are constants $C_{\delta,k}$ depending only on k and δ so that

$$\|\nabla^k \phi\|_{g_0}(y) \leq C_{\delta,k} r(y)^{-2n+2-k+\delta}, \quad \textit{for } , y \in Y_{r_0}. \tag{22}$$

Consider the elliptic operator P defined by

$$(Pu)\omega_0^n := i\partial\bar{\partial}u \wedge (\omega^{n-1} + \omega^{n-2}\omega_0 + \cdots + \omega_0^{n-1}). \tag{23}$$

On Y_{r_0}, $g_0 = dr^2 + r^2 g$, the cone metric, and the Euler vector field $r\partial_r$ generates an action of $\mathbb{R}_{>0}$ by homothetic isometries on g_0. For $a > 1$ denote this action by $\psi_a : Y_{r_0} \to Y_{r_0}$.

$$\psi_a^* g_0 = a^2 g_0. \tag{24}$$

Then we have a weighted version of the schauder estimates for $P(u) = w$

$$\|u\|_{C^{k+2,\alpha}_{\beta}} \leq C \left(\|w\|_{C^{k,\alpha}_{\beta-2}} + \|u\|_{C^0_{\beta}} \right), \tag{25}$$

where $C^{k,\alpha}_{\beta}$ is the *weighted* Hölder space. Apply (25) to $P(\phi) = e^f - 1$ with $\beta = -2n + 2 + \delta$.

Asymptotics

Proposition

Let g be the Ricci-flat Kähler metric on Y of Proposition 3.1. Then curvature of g satisfies

$$\|\nabla^k R(g)\|_g = O(r^{-2-k}), \quad \text{for } k \geq 0. \tag{26}$$

Furthermore, if $\|R(g)\|_g = O(r^{-\alpha})$*, for* $\alpha > 2$*, then* (Y, g) *is asymptotically locally Euclidean.*

The last statement is due to S. Bando, A. Kasue, and H. Nakajima, '89.

Uniqueness

We consider the uniqueness of the metric in Theorem (1.2).

Proposition

Let g be the Ricci-flat metric of Theorem (1.2) and g_1 another Ricci-flat Kähler metric with $[\omega_1] = [\omega]$ and $|g_1 - g| \in C^{0,\alpha}_{\beta}$, $\beta < -n$. Then $g_1 = g$.

Note: In Theorem (1.2) $|g - g_0| \in C^{\infty}_{-2n+\delta}$.

Lemma

For each $f \in C^{k,\alpha}_{\beta}(Y)$, $-2n < \beta < -2$, there is a unique $u \in C^{k+2,\alpha}_{\beta+2}(Y)$ with $\Delta u = f$.

It is well known that

$$\Delta : L^2_{k+2,\delta+2}(Y) \to L^2_{k,\delta}, \quad \text{is Fredholm for } \delta \text{ outside a discrete “exceptional set"} \quad (27)$$

Then the elliptic maximum principle and integration by parts shows that (27) is an isomorphism for δ non-exceptional. So we have $u \in L^2_{k+2,\delta+2}(Y)$ for some $-2 > \delta > \beta$.
Applying the maximum priciple to $C\rho^{\beta}$, as $\Delta(\rho^{\beta+2}) = O(\rho^{\beta}), \leq 0$, and Schauder estimates shows $u \in C^{k+2,\alpha}_{\beta+2}(Y)$.

Uniqueness

Lemma ($\partial\bar{\partial}$-Lemma)

Suppose η is a smooth exact $(1,1)$-form and $\eta \in C^{0,\alpha}_{\beta}(\Lambda^{1,1}(Y))$, $\beta < -n$. Then there is a smooth function $f \in C^{2,\alpha}_{\beta+2}(Y)$ with $\eta = dd^c f$.

Proof.

For Proposition 3.5, $\eta = \omega_1 - \omega$ is exact. So Lemma 3.7 $\eta = dd^c u$, $u \in C^{2,\alpha}_{\beta+2}(Y)$. Then apply the maximum princlple to the operator

$$P(u)\omega^n := dd^c u \wedge \sum_{j=0}^{n-1} \omega_1^j \wedge \omega^{n-1-j} = 0. \tag{31}$$

Toric resolutions

As an algebraic variety $X = X_\Delta$ where Δ is the fan in $\mathbb{Z}_T \cong \mathbb{Z}^n$ defined by the dual cone $\mathcal{C}(\mu)^*$, spanned by $u_1, \dots, u_d \in \mathbb{Z}_T$.
We assume that X is Gorenstein. Thus there is a $\gamma \in \mathbb{Z}_T^*$ so that $\gamma(u_i) = -1$ for $i = 1, \dots, d$.

$$P_\Delta := \{x \in \mathcal{C}(\mu)^* : \langle \gamma, x \rangle = -1\} \subset H_\gamma \cong \mathbb{R}^{n-1} \quad (32)$$

A toric crepant resolution

$$\pi : X_{\tilde{\Delta}} \to X_\Delta \quad (33)$$

is given by a nonsingular subdivision $\tilde{\Delta}$ of Δ with every 1-dimensional cone $\tau_i \in \tilde{\Delta}(1), i = 1, \dots, N$ generated by a primitive vector $u_i := \tau_i \cap H_\gamma$.
This is equivalent to a *basic* lattice triangulation of P_Δ.

- *Lattice* means that the vertices of every simplex are lattice points.
- A triangulation is *maximal* if vertices of simplices are its only lattice points.
- *basic* means that the vertices of every top dimensional simplex generates a basis of $\mathbb{Z}^{n-1}$.
- When $n = 3$, P_Δ is 2-dimensional and every maximal triangulation is basic.

Kähler structures

A Kähler structure on $X_{\tilde{\Delta}}$ is given by a strictly convex support function $h \in \mathrm{SF}(\tilde{\Delta})$.
For each $\tau_j \in \tilde{\Delta}(1)$ we have a primitive element $u_j \in \mathbb{Z}_T, j = 1, \ldots, N$.
Set $\lambda_i := h(u_i)$. Then define the rational convex polyhedral set

$$\mathcal{C}_h := \bigcap_{j=1}^{N} \{y \in \mathfrak{t}^* : \langle u_j, y \rangle \geq \lambda_j\}. \quad (34)$$

We employ a construction originally due to Delzant and extended to the non-compact and singular cases by D. Burns, V. Guillemin, and E. Lerman which constructs a Kähler structure on $X_{\tilde{\Delta}}$ associated to a convex polyhedral set $\mathcal{C}_h$.

Kähler structures

Let $\mathcal{A} : \mathbb{Z}^N \to \mathbb{Z}_T$ be the $\mathbb{Z}$-linear map with $\mathcal{A}(e_i) = u_i$, where $e_i, i = 1, \ldots, N$ is the standard basis of $\mathbb{Z}^N$.
$\mathcal{A}$ induces a map of Lie algebras $\mathcal{A} : \mathbb{R}^N \to \mathfrak{t}$. Let $\mathfrak{k} = \ker \mathcal{A}$.
We have an exact sequence

$$0 \to \mathfrak{k} \xrightarrow{\mathcal{B}} \mathbb{R}^N \xrightarrow{\mathcal{A}} \mathfrak{t} \to 0. \qquad (35)$$

And the dual

$$0 \to \mathfrak{t}^* \xrightarrow{\mathcal{A}^*} (\mathbb{R}^N)^* \xrightarrow{\mathcal{B}^*} \mathfrak{k}^* \to 0. \qquad (36)$$

Also $\mathcal{A}$ induces a surjective map of Lie groups $\bar{\mathcal{A}} : T^N \to T^n$, where $T^N = \mathbb{R}^N / 2\pi\mathbb{Z}^N$. If
$K = \ker \bar{\mathcal{A}}$, then we have the exact sequence

$$1 \to K \longrightarrow T^N \xrightarrow{\bar{\mathcal{A}}} T^n \to 1. \qquad (37)$$

Kähler structures

The moment map Φ for the action of T^N on $(\mathbb{C}^N, \frac{\mathrm{i}}{2}\sum_{j=1}^N dz_j \wedge d\bar{z}_j)$ is

$$\Phi(z) = \sum_{j=1}^{N} |z_j|^2 e_j^*. \tag{38}$$

Then moment map Φ_K for the action of K on $\mathbb{C}^N$ is the composition

$$\Phi_K = \mathcal{B}^* \circ \Phi. \tag{39}$$

Let $\lambda = \sum_{j=1}^N \lambda_j e_j^*$, and $\nu = \mathcal{B}^*(-\lambda)$. Then

$$M_{\mathcal{C}_h} := \Phi_K^{-1}(\nu)/K \tag{40}$$

is smooth provided $\mathcal{C}_h$ in non-singular as a polyhedron.
As complex toric varieties $M_{\mathcal{C}_h} \cong X_{\tilde{\Delta}}$.

Kähler structures

Suppose $\tilde{\Delta}$ is a nonsingular subdivision of Δ giving a crepant resolution.
Then $u_1, \ldots, u_d \in \mathbb{Z}_T$ are vectors spanning the cone $\mathcal{C}^*(\mu)$, whereas $u_{d+1}, \ldots, u_N \in \mathbb{Z}_T$ are the lattice points in $\overset{\circ}{P}_\Delta$.
We want a Kähler form ω on $X_{\tilde{\Delta}}$ with $[\omega] \in H^2_c(X_{\tilde{\Delta}}, \mathbb{R})$ so we make the following definition.

Definition

A strictly convex support function $h \in \mathrm{SF}(\tilde{\Delta}, \mathbb{R})$ is compact if $h(u_j) = 0$ for $j = 1, \ldots, d$.

Kähler structures

The moment map for $T^n = T^N/K$ acting on $M_{\mathcal{C}_h}$ is

$$\Phi_{\mathcal{C}_h} : M_{\mathcal{C}_h} \to \mathfrak{t}^* \tag{41}$$

If $l_j(y) = \langle u_j, y\rangle - \lambda_j$ for $j = 1, \ldots, N$, and $l_\infty(y) = \sum_{j=1}^N \langle u_j, y\rangle$, we have

Theorem (V. Guilleimin)

The Kähler form ω_h on the preimage $\Phi_{\mathcal{C}_h}^{-1}(\mathring{\mathcal{C}}_h)$ of the interior $\mathring{\mathcal{C}}_h$ of the polyhedral set $\mathcal{C}_h$ is

$$\omega_h = \mathbf{i}\partial\bar{\partial}\Phi_{\mathcal{C}_h}^*\Big(\sum_{j=1}^N \lambda_j \log(l_j) + l_\infty\Big).$$

Notice that the potential is singular only on the exceptional set.

3-dimensional toric varieties

Proposition

Let $X = X_\Delta$ be a 3-dimensional Gorenstein toric cone variety. Suppose $\overset{\circ}{P}_\Delta$ contains a lattice point, i.e. X is not a terminal singularity. Then there is a basic lattice triangulation of P_Δ such that the corresponding subdivision $\tilde{\Delta}$ admits a compact strictly upper convex support function $h \in \mathrm{SF}(\tilde{\Delta}, \mathbb{R})$.

This follows by making generalized blow-ups at points, and along curves, at each lattice point in $\overset{\circ}{P}_\Delta$. The support function h is defined inductively. It ends with a maximal triangulation of P_Δ which is basic because P_Δ is 2-dimensional.

3-dimensional toric varieties

Theorem

Let X be a three dimensional Gorenstein toric Kähler cone with an isolated singularity which is not the quadric hypersurface, as a variety. Then there is a crepant resolution $\pi : Y \to X$ such that Y admits a Ricci-flat Kähler metric g which is asymptotic to $(C(S), \bar{g})$ as in (1). Furthermore, g is invariant under the compact torus T^3.

Infinite series of toric Sasaki-Einstein 5-manifolds have been constructed using Theorem 2.7 by K. Cho, A. Futaki, and H. Ono. Together with the series $S^{p,q}$ we

Theorem

For each $m \geq 1$, exists infinitely many toric asymptotically conical Ricci-flat Kähler manifolds Y asymptotic to a cone over a Sasaki-Einstein structure on $\#m(S^2 \times S^3)$. For each $m \geq 1$, the Betti numbers, $b_2(Y) = m + c(X)$, $b_4(Y) = c(X)$, of the Y become arbitrarily large.

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Resolutions of $C(S^{p,q})$

A series of 5-dimensional Sasaki-Einstein metrics $S^{p,q}$, with $p, q \in \mathbb{N}, p > q > 0$, and $\gcd(p, q) = 1$ are due to J. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, 2004. They contain the first known examples of irregular Sasaki-Einstein, and also are given explicitly. These examples are toric and are further of cohomogeneity one with an isometry group of $SO(3) \times U(1) \times U(1)$ if p, q are both odd, and $U(2) \times U(1)$ otherwise.
The Sasaki structure is quasi-regular precisely when $p, q \in \mathbb{N}$ as above satisfy the diophantine equation

$$4p^2 - 3q^2 = r^2, \tag{42}$$

for some $r \in \mathbb{Z}$.

We have $X_\Delta = C(S^{p,q}) \cup \{o\}$ where the fan Δ in $\mathbb{Z}^3$ is generated by the four vectors

$$u_1 = (0,0,1), u_2 = (1,0,1), u_3 = (p,p,1), u_4 = (p-q-1, p-q, 1). \qquad (43)$$

A basic lattice triangulation of P_Δ can be constructed for general p, q as is shown in Figure 1 for $S^{5,3}$. It is not difficult to see that the subdivision $\tilde{\Delta}$ of Δ has a compact strictly convex support function. Thus Corollary 1.3 gives a $p-1$-dimensional family of asymptotically conical Ricci-flat Kähler metrics on $X_{\tilde{\Delta}}$.

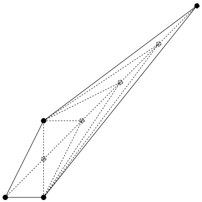

Figure: A resolution of $X^{5,3}$

Hypersurface singularities

We will now consider weighted homogeneous hypersurface singularities. Let $\mathbf{w} = (w_0, \dots, w_n) \in (\mathbb{Z}_+)^{n+1}$ with $\gcd(w_0, \dots, w_n) = 1$. We have the weighted $\mathbb{C}^*$-action $\mathbb{C}^*(\mathbf{w})$ on $\mathbb{C}^{n+1}$ given by $(z_0, \dots, z_n) \to (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n)$
A polynomial $f \in \mathbb{C}[z_0, \dots, z_n]$ is weighted homogeneous of degree $d \in \mathbb{Z}_+$ if

$$f(\lambda^{w_0} z_0, \cdots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \dots, z_n). \tag{44}$$

Assume that the origin is an isolated singularity. So the link

$$S_f = X_f \cap S^{2n+1}, \tag{45}$$

is a smooth $(2n-1)$-dimensional manifold.
If $f \in \mathbb{C}[z_0, \dots, z_n]$ is quasi-homogeneous, then we have the hypersurface in the weighted projective space

$$Z_f := \{[z_0 : \cdots : z_n] : f(z_0, \dots, z_n) = 0\} \subset \mathbb{C}P(w_0, \dots, w_n). \tag{46}$$

Hypersurface singularities

We have

$$\begin{array}{ccc} S_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \downarrow \pi & & \downarrow \\ Z_f & \longrightarrow & \mathbb{C}P(\mathbf{w}) \end{array}$$

And S_f has a Sasakian structure by restricting a weighted structure on S^{2n+1}

Proposition

The orbifold Z_f is Fano, i.e. the orbifold canonical bundle $\mathbf{K}_{Z_f}$ is negative, if and only if $|\mathbf{w}| = \sum_{j=0}^{n} w_j > d$.

It follows that the cone $C(S_f)$ satisfies the condition of Proposition 2.3. In fact, by the adjunction formula the n-forms

$$\Omega_k := \frac{(-1)^k}{\partial f / \partial z_k} dz_0 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_n|_X, \qquad (47)$$

glue together to a global generator of the canonical bundle $\mathbf{K}_{X_f}$.

Resolutions of hypersurfaces

The *weighted blow-up* generalizes the usual blow-up.
Let $\mathbf{w} = (w_0, \ldots, w_n)$ be a weight vector.
Then $S(\mathbf{w}) = \mathbb{C}[z_0, \ldots, z_n]$ has a corresponding grading. So $S(\mathbf{w}) = \sum_{j \geq 0} S(j)$, where $S(j)$ are the homogeneous elements of degree j.
$f \in S(\mathbf{w})$ is written in homogeneous components $f = \sum_{j \geq 0} f(j)$, then we define the degree of f to be $w(f) = \min_{j \geq 0}\{f(j) \neq 0\}$.
We have ideals $M^{\mathbf{w}}(j) = \{f \in S(\mathbf{w}) : w(f) \geq j\}$.

Definition

Then the weighted blow-up $B_0^{\mathbf{w}}\mathbb{C}^{n+1}$ *of* $\mathbb{C}^{n+1}$ *with weight* $\mathbf{w}$ *is* $\mathrm{Proj}(\sum_{j \geq 0} M^{\mathbf{w}}(j))$.

Geometrically, $B_0^{\mathbf{w}}\mathbb{C}^{n+1}$ is the total space of the tautological line V-bundle over $\mathbb{C}P(\mathbf{w})$ associated to the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus \{0\}$, which has associated rank 1 sheaf $\mathcal{O}(-1)$.
For any variety $X \subset \mathbb{C}^{n+1}$ the weighted blow-up $X' = B_0^{\mathbf{w}}X$ is the birational transform of X in $B_0^{\mathbf{w}}\mathbb{C}^{n+1}$.
We have the adjunction formula, E is the exceptional divisor,

$$K_{X'} = \pi^* K_X + (w(z_0 \cdots z_n) - w(f) - 1)E|_{X'}. \qquad (48)$$

Examples with $b_3 \neq 0$

X	S S-E	crepant Y		$c(X)$	$b_3(Y)$	$H_2(S)$
$x_0^3 + x_1^3 + x_2^3 + x_3^k = 0$	$k = 3$, $k > 6$	0	yes	$\lfloor \frac{k}{3} \rfloor$	$2(\lfloor \frac{k}{3} \rfloor - 1)$	$\mathbb{Z}^6 \oplus \mathbb{Z}^2_{\frac{k}{3}}$
		1	yes	$\lfloor \frac{k}{3} \rfloor$	$2\lfloor \frac{k}{3} \rfloor$	$\mathbb{Z}^2_k$
		2	no			$\mathbb{Z}^2_k$
$x_0^2 + x_1^4 + x_2^4 + x_3^k = 0$	$k = 4$, $k > 10$	0	yes	$\lfloor \frac{k}{4} \rfloor$	$2(\lfloor \frac{k}{4} \rfloor - 1)$	$\mathbb{Z}^7 \oplus \mathbb{Z}^2_{\frac{k}{4}}$
		1	yes	$\lfloor \frac{k}{4} \rfloor$	$2\lfloor \frac{k}{4} \rfloor$	$\mathbb{Z}^2_k$
		2	unknown			$\mathbb{Z}^3 \oplus \mathbb{Z}^2_{\frac{k}{2}}$
		3	no			$\mathbb{Z}^2_k$
$x_0^2 + x_1^3 + x_2^6 + x_3^k = 0$	$k = 6$, $k > 12$	0	yes	$\lfloor \frac{k}{6} \rfloor$	$2(\lfloor \frac{k}{6} \rfloor - 1)$	$\mathbb{Z}^8 \oplus \mathbb{Z}^2_{\frac{k}{6}}$
		1	yes	$\lfloor \frac{k}{6} \rfloor$	$2\lfloor \frac{k}{6} \rfloor$	$\mathbb{Z}^2_k$
		2	unknown			$\mathbb{Z}^2 \oplus \mathbb{Z}^2_{\frac{k}{2}}$
		3	unknown			$\mathbb{Z}^4 \oplus \mathbb{Z}^2_{\frac{k}{3}}$
		4	unknown			$\mathbb{Z}^2 \oplus \mathbb{Z}^2_{\frac{k}{2}}$
		5	no			$\mathbb{Z}^2_k$
$x_0^3 + x_1^4 + x_2^4 + x_3^4 = 0$	yes	yes		3	12	$\mathbb{Z}^6_3$

- The 3 series are resolved by succesively blowing up with weights $(1,1,1,1),(2,1,1,1),(3,2,1,1)$ respectively.
- The exceptional divisors are elliptic ruled surfaces.
- The existence of the Sasaki-Einstein metric on the link S is due to C. Boyer, K. Galicki, and J. Kollár, 2003.
- For values $k=3,4,6$ in the second column Z_f is just the del Pezzo surface of degree $3,2$ and 1, respectively.
- The last example is a $(4,3,3,3)$ blow-up, then a blow-up along a genus 3 curve.

Higher diminsional example

X	S S-E	$k \mod n$	$c(X)$
$x_0^n + x_1^n + \cdots + x_{n-1}^n + x_n^k = 0$	$k > n(n-1)$, $k = n$	0	$\lfloor \frac{k}{n} \rfloor$
		1	$\lfloor \frac{k}{n} \rfloor$

The exceptional divisors of the resolution $\pi : Y_k \to X_k$ are ruled varieties $E_j = \mathbb{P}(\mathcal{O}_F(1) \oplus \mathcal{O}_F), j = 1, \dots, c(X) - 1$, besides the last which for $k = 0 \mod n$ is the Fano hypersurface $E_c = \{x_0^n + x_1^n + \cdots + x_{n-1}^n + x_n^n = 0\} \subset \mathbb{C}P^n$ and for $k = 1 \mod n$ is the cone over F $E_c = \{x_0^n + x_1^n + \cdots + x_{n-1}^n = 0\} \subset \mathbb{C}P^n$.
$F = \{x_0^n + \cdots + x_{n-1}^n = 0\} \subset \mathbb{C}P^{n-1}$, the Calabi-Yau Fermat hypersurface.

Question

Question

Do there exist any examples which are asymptotic to a cone over a topological sphere, $S \underset{homeo}{\cong} S^{2n-1}$.