Einstein metrics and exotic smooth structures on 4-manifolds

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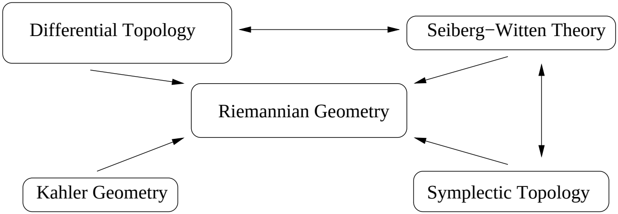
Main question

Question

What is the relation between the existence of Einstein metrics and the differential structure considered on a $4-$*manifold?*

Main ingredients

(M, g) compact, oriented, smooth $4-$manifold, g a Riemannian metric



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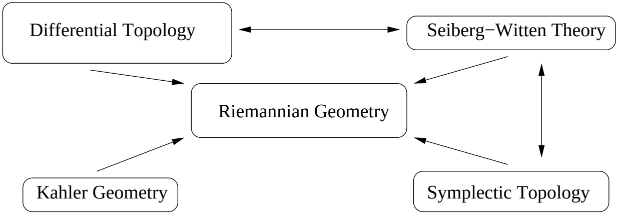

Topological invariants

- fundamental group: $\pi_1(M)$
- Second Stiefel-Whitney class: $w_2(M) \in H^2(M, \mathbb{Z}_2)$
- Signature: $\tau(M) = b^+ - b^-$
- Euler Characteristic: $\chi(M)$
- Freedman, Donaldson: Compact, smooth, simply connected 4-manifolds are classified, up to homeomorphism, by their topological invariants: $\chi(M), \tau(M)$, and the parity of the intersection form (i.e $w_2 = 0$ or $\neq 0$).
- Consequence: Any simply connected, non-spin ($w_2 \neq 0$) manifold is homeomorphic to $a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$.
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$$(2\chi \pm 3\tau)(M) = \frac{1}{4\pi^2} \int_M \left[\frac{s^2}{24} + 2|W^\pm|^2 - \frac{|\overset{\circ}{r}|^2}{2} \right] d\mu_g$$

where $s, W^\pm, \overset{\circ}{r}$ are the scalar, Weyl, trace free Ricci curvatures and μ_g the volume form

Theorem (Hitchin-Thorpe Inequality)

If the smooth compact oriented 4-manifold M admits an Einstein metric g, then

$$(2\chi \pm 3\tau)(M) \geq 0,$$

with equality if (M, g) is finitely covered by a flat 4-torus T^4 or by the $K3$ surface with a hyper-Kähler metric or by the orientation-reversed version of $K3$ with a hyper-Kähler metric.

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Seiberg-Witten Theory

Given (M, g) and let $\mathbb{V}_{\pm}$ be the spin^c structure associated to the Hermitian line bundle $L, (c_1(L) \equiv w_2(M) \mod 2)$.
The Seiberg-Witten Equations:

$$D_A \Phi = 0 \quad (1)$$
$$F_A^+ = i\sigma(\Phi). \quad (2)$$

where $\Phi \in \Gamma(\mathbb{V}_+)$, A a connection on L, F_A^+ is the self-dual part of the curvature of A, and where $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map satisfying

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

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The Seiberg-Witten Invariant, $SW_g(L)$: the number of solutions, (A, Φ), of a generic perturbation of the Seiberg-Witten monopole equation, modulo gauge transformation and counted with orientations.

- If $b^+(M) \geq 2$, the Seiberg-Witten invariant is a diffeomorphism invariant, i.e. independent of the metric g.

- There are large classes of manifolds for which the invariant is non-trivial: symplectic manifolds, manifolds obtained via gluing. (Taubes, Szabó, Morgan, etc.)
- Weitzenböck formula for the Dirac operator D_A in relation with the Seiberg-Witten equations:

$$0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4$$

In particular, there are no positive scalar curvature metrics on manifolds with non-trivial S-W invariant.

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A differential obstruction to existence of Einstein metrics

Theorem (LeBrun '01)

Let X be a compact oriented 4-manifold with a non-trivial Seiberg-Witten invariant and with $(2\chi + 3\tau)(X) > 0$. Then

$$M = X \# k\overline{\mathbb{CP}^2} \# l(S^1 \times S^3)$$

does not admit Einstein metrics if $k + 4l \geq \frac{1}{3}(2\chi + 3\tau)(X)$.

Key ingredient: curvature estimates:

$$\frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|)^2\right)d\mu \geq \frac{2}{3}(c_1^+(L))^2$$

where c_1^+ is the self-dual part of $c_1(L)$.

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Small Topology: $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$

Question (Besse)

Is the sign of the Einstein determined by the homeomorphism class of the manifold?

Answer: No. Catanese-LeBrun 1997.
Example: $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$ and the Barlow surface (complex surface of general type, with *ample* canonical line bundle).

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Theorem (Rasdeaconu, S. '08)

Each of the topological 4-manifolds $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, for $k = 5, 6, 7, 8$ admits a smooth structure which has an Einstein metric of scalar curvature $s > 0$, a smooth structure which has an Einstein metric with $s < 0$ and infinitely many non-diffeomorphic smooth structures which do not admit Einstein metrics.

On $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, $k = 1, \dots, 8$ with the canonical smooth structures the existence of a positive scalar curvature Einstein metric was proved by Page ($k = 1$), Chen-LeBrun-Weber ($k = 2$), Siu, Tian-Yau ($k \geq 3$).

- On $M = \mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, $k = 5, \ldots, 8$, with Rasdeaconu, we show that the exotic complex structures constructed by Park and collaborators ('07, '08), have ample canonical line bundle. Hence they admit a Kähler-Einstein metrics of negative scalar curvature by Calabi-Yau conjecture.
- One expects to use the same methods to obtain negative curvature Einstein metrics on exotic smooth structures for smaller k.
- Starting with exotic smooth structures on $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$, (due to Akhmedov, Baykur and Park '07), we construct infinitely many exotic smooth structures on M which don't admit an Einstein metric. All these exotic smooth structures have negative Yamabe invariant.
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Non-existence theorem

Theorem (S.)

For any small $\epsilon > 0$ there exists an $N(\epsilon) > 0$ such that for any integer $d \geq 2$ and any integer lattice point (n, m), satisfying:

- $n > 0$
- $d/n, d/m$
- $n < (6-\epsilon)m - N(\epsilon)$

there exist infinitely many free, non-equivalent smooth $\mathbb{Z}/d\mathbb{Z}-$actions on $M = (2m-1)\mathbb{CP}^2\#(10m-n-1)\overline{\mathbb{CP}^2}$ (i.e $(2\chi+3\tau)(M) = n, \frac{\chi+\tau}{4}(M) = m$). Moreover, there is no Einstein metric on M invariant under any of the $\mathbb{Z}/d\mathbb{Z}-$actions.

- Hitchin-Thorpe inequality: $n > 0$
- Admissibility condition: $d/n, d/m$

Region: $n < (6-\epsilon)m - N(\epsilon)$ determined by the geography of simply connected, symplectic manifolds due to Braungardt, Kotschick (2005).

If we denote by $\Gamma_i, i \in \mathbb{N}$, the actions of $\mathbb{Z}/d\mathbb{Z}$ on M, then the quotient manifolds M/Γ_i are homeomorphic but mutually non-diffeomorphic.

$$M/\Gamma_i = Y_i\#k\overline{\mathbb{CP}^2}\#S_d$$

- Y_i are homeomorphic, non-diffeomorphic, simply connected, symplectic 4−manifolds,
- S_d a rational homology sphere, $\pi_1(S_d) = \mathbb{Z}_d$, $\widetilde{S_d} = \#(d-1)(S^2 \times S^2)$.

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- M has trivial Seiberg-Witten invariant, but the M/Γ_i has non-trivial solutions of the S-W equations.
- The Yamabe invariant of M $(= a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2})$ is positive, while if we consider the Yamabe invariant of the conformal class of a $\mathbb{Z}_d$ invariant metric g, $Y_{[g]} < 0$.
- Infinitely many other actions can be exhibited on M.
- The results in the above theorem are stated for finite cyclic groups, but they also hold for groups acting freely on the 3-dimensional sphere or for direct sums of the above groups.

Proposition (S.)

On $M = 15\mathbb{CP}^2\#77\overline{\mathbb{CP}^2}$, there exists an involution σ, acting freely on the manifold, such that $15\mathbb{CP}^2\#77\overline{\mathbb{CP}^2}$ does not admit an Einstein metric invariant under the involution σ.

See that: $n = (2\chi + 3\tau)(M) = 2, m = \frac{\chi+\tau}{4}(M) = 8.$

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Existence theorem

Theorem (S.)

There are infinitely many compact, smooth, simply connected, non-spin manifolds $M_i, i \in \mathbb{N}$, whose topological invariants verify $(2\chi + 3\tau)(M_i) = n > 0, (2\chi + 3\tau)(M_i) < 5(\frac{\chi+\tau}{4})(M_i))$, and satisfy the following conditions:

- *There is at least one free, smooth, $\mathbb{Z}/d\mathbb{Z}$ action on M_i,*
- *M_i admits an Einstein metric which is invariant under the above $\mathbb{Z}/d\mathbb{Z}$ action,*
- *M_i is not diffeomorphic to $M_{can} = (2m-1)\mathbb{CP}^2\#(10m-n-1)\overline{\mathbb{CP}^2}$, but $M_i\#\mathbb{CP}^2$ and $M_{can}\#\mathbb{CP}^2$ are diffeomorphic.*

- M_i are complex surfaces with ample canonical line bundle, and admit Kähler-Einstein metrics.
- Construct $M_i = M$ as an iterated cyclic branched cover:

$$M \xrightarrow{\pi_2} N \xrightarrow{\pi_1} \mathbb{CP}^1 \times \mathbb{CP}^1$$

π_1 is a $d-1$ cover, branched along D, s.t $\mathcal{O}(D) = \mathcal{O}(da, db)$
π_2 is a $p-1$ cover, branched along $\pi_1^{-1}(C)$, s.t
$\mathcal{O}(C) = \mathcal{O}(pm, pn)$

$$M \subset \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(a, b) \oplus \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(m, n)$$

- The $\mathbb{Z}_d$ action on $\mathbb{CP}^1 \times \mathbb{CP}^1 : \rho^d = 1, \rho([z_1 : z_2]) = [\rho z_1 : z_2]$ extends to M if the defining polynomials for D, C are $\mathbb{Z}_d-$invariant

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There is a dictionary between the properties of M and the numerical data: a, b, m, n.

- $D^2 \neq 0, C^2 \neq 0 \Longrightarrow M$ simply connected
- $$\begin{matrix} (d-1)a + (p-1)m - 2 > 0 \\ (d-1)b + (p-1)n - 2 > 0 \end{matrix} \Longrightarrow K_M \text{ ample}$$
- $a+1, b+1, a+b+1$ relatively prime to $d \Longrightarrow$ there exists a free holomorphic $\mathbb{Z}_d$ action on M

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Proposition

The iterated branched cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$, branched along pull-backs of positive self-intersection curves, transverse to each other, is almost completely decomposable.

Idea of proof: Use double induction on the number of branched covers and the degree of the last cover, and Mandelbaum Moishezon techniques (1980) on normal crossing degenerations of the manifold.

Spin manifolds: an obstruction theorem

Theorem (S.)

There exists an integer $n_0 > 0$ such that for any integer $d > n_0$ the manifolds:

1. $M_{1,n} = d(n+5)(K3)\#(d(n+7)-1)(S^2 \times S^2)$
2. $M_{2,n} = d(2n+5)(K3)\#(d(2n+6)-1)(S^2 \times S^2)$

$n \in \mathbb{N}^*$, admit infinitely many non-equivalent free $\mathbb{Z}/d\mathbb{Z}$ actions, such that there is no Einstein metric on $M_{1,n}, M_{2,n}$ invariant under any of the $\mathbb{Z}/d\mathbb{Z}-$actions.

$$M_{1,n}/\Gamma_j = X\#K3_{(2j+1)}\#E(2n)\#S_d$$

where X is a smooth hypersurface of tridegree $(4,4,2)$ in $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, $(c_1^2(X) = 16,\ c_2(X) = 104,\ b_2^+(X) = 19)$.

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Higher dimensional manifolds

Question

Is the sign of the Einstein metric determined by the diffeomorphism class of the manifold?

Proposition

Let $N_1 = \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$, $N_2 = \mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$, $N_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$ and $N_4 = \mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. Then the smooth manifold N obtained by taking the $k-$fold products, $k \geq 2$, of arbitrary N_1, N_2, N_3 or N_4, admits two Einstein metrics g_1, g_2 such that the signs of the scalar curvature are $s_{g_1} = -1, s_{g_2} = +1$. Moreover, these metrics are Kähler-Einstein with respect to two distinct complex structures J_1, J_2.

Remark: g_1, g_2 Kähler metrics $\Longrightarrow Vol_{g_1}(N) = Vol_{g_2}(N)$

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Is the sign of the Einstein metric determined by the diffeomorphism class of the manifold?

Proposition

Let $N_1 = \mathbb{CP}^2 \# 8\overline{\mathbb{CP}^2}$, $N_2 = \mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$, $N_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$ *and* $N_4 = \mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. *Then the smooth manifold* N *obtained by taking the* $k-$*fold products,* $k \geq 2$, *of arbitrary* N_1, N_2, N_3 *or* N_4, *admits two Einstein metrics* g_1, g_2 *such that the signs of the scalar curvature are* $s_{g_1} = -1, s_{g_2} = +1$. *Moreover, these metrics are Kähler-Einstein with respect to two distinct complex structures* J_1, J_2.

Remark: g_1, g_2 Kähler metrics $\Longrightarrow Vol_{g_1}(N) = Vol_{g_2}(N)$

Thank you!