Generalized Contact Structures

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The bundle $TM \oplus T^*M \to M$

- A symmetric bilinear form $\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(\alpha(Y)+\beta(X))$.
- Courant bracket

$$[\![X+\alpha, Y+\beta]\!] = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha).$$

Remarks:

- $\langle -, - \rangle$ is non-degenerate.
- TM and T^*M are maximally isotropic. $\langle X, Y\rangle = 0$, $\langle \alpha, \beta\rangle = 0$ for all X, Y, α, β.
- Courant bracket does not satisfy Jacobi identity.

Lemma

If V is a subbundle of $(TM \oplus T^*M)_{\mathbb{C}}$ such that its space of sections is closed: $[\![v_0, v_1]\!] \in C^\infty(M, V)$, and if V is isotropic: $\langle v_0, v_1\rangle = 0$, for any sections v_0 and v_1 of V, with $\rho : V \hookrightarrow (TM \oplus T^*M)_{\mathbb{C}} \to TM_{\mathbb{C}}$, then the triple $(V, [\![-,-]\!]_V, \rho)$ is a Lie algebroid.

Lie bialgebroids, Liu-Weinstein-Xu (90's)

Definition

L and K form a Lie bialgebroid pair in $TM \oplus T^*M$ if

- L and K are maximally isotropic with respect to $\langle -, - \rangle$,
- $L \oplus K = TM \oplus T^*M$;
- (space of sections of) L and K are closed under $[\![-, -]\!]$.
- When $d_K[\![\ell_1, \ell_2]\!] = [\![d_K\ell_1, \ell_2]\!] + [\![\ell_1, d_K\ell_2]\!]$, where

$$(d_K\ell)(k_1, k_2) := 2\Big(\rho(k_1)\langle \ell, k_2 \rangle - \rho(k_2)\langle \ell, k_1 \rangle - \langle \ell, [\![k_1, k_2]\!] \rangle\Big).$$

Treat L as K^*, $d_K : \wedge^m L \to \wedge^{m+1} L$.

Deformation of Lie bialgebroids

Suppose that (L, K) is a Lie bialgebroid. $L \oplus K = (TM \oplus T^*M)_{\mathbb{C}}$.
Let $\Gamma \in C^\infty(M, \wedge^2 L) \subset C^\infty(M, \mathrm{Hom}(L^*, L)) = C^\infty(M, \mathrm{Hom}(K, L))$.
Let K_Γ be the graph of K with respect to Γ:

$$K_\Gamma = \{k + \Gamma(k) : k \in C^\infty(M, K)\}.$$

$$L \oplus K_\Gamma \cong L \oplus K \cong L \oplus L^*. \quad K_\Gamma \subset L \oplus K.$$

Theorem (LWX)

(L, K_Γ) is a Lie bialgebroid pair if and only if $d_K \Gamma + \frac{1}{2}[\![\Gamma, \Gamma]\!] = 0$.

$[\![-, -]\!]$ on $\wedge^\bullet L$, d_K is C-E differential of $[\![-, -]\!]$ on $\wedge^\bullet K$.

Generalized complex structures

Definition

A generalized almost complex structure on an even-dimensional manifold M is a bundle automorphism $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ such that $\mathcal{J}^2 = -\mathbb{I}$ and $\mathcal{J}^* + \mathcal{J} = 0$.

$$\mathcal{J} = \begin{pmatrix} \varphi & \pi \\ \theta & -\varphi^* \end{pmatrix},$$

φ a (1,1)-tensor, π a bivector field, θ a 2-form.

$$(TM \oplus T^*M)_{\mathbb{C}} = L \oplus \overline{L} = +\text{i eigenspace} \oplus \text{ -i eigenspace}$$

Equivalent definition: choice of maximally isotropic subspace L in $(TM \oplus T^*M)_{\mathbb{C}}$ as $(+$i) eigenspace. The dual space L^* as (-i) eigenbundle.

Integrability

Definition

$\mathcal{J}$ is integrable if $C^\infty(M, L)$ and/or $C^\infty(M, \overline{L})$ are closed with respect to $[\![-,-]\!]$.

When $(M, \mathcal{J})$ is a generalized complex structure,

$$L \oplus \overline{L} = (TM \oplus T^*M)_{\mathbb{C}}, \quad \overline{L} \cong L^*.$$

In particular, the pair $(L, \overline{L})$ forms a Lie bialgebroid.

Examples

(1) When $J : TM \to TM$ is a (classical) complex structure. On $TM \oplus T^*M$, define

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix},$$

$L = T^{1,0} \oplus T^{*(0,1)}$.

(2) θ is a symplectic form. π Poisson (bi)vector field. Define

$$\mathcal{J} = \begin{pmatrix} 0 & \pi \\ \theta & 0 \end{pmatrix}.$$

$L = \mathrm{Span}\{X - i\iota_X\theta : X \in C^\infty(M, TM)\}$.
Integrability of L and $\overline{L}$ is equivalent to $d\theta = 0$.

Generalized Deformation. Gaultieri 04

1. Given $(M, \mathcal{J})$ a classical complex structure.
2. Treat it as generalized complex structure.
3. Construct the Lie bialgebroid: $L \oplus \overline{L}$.
4. Given Γ_1 in $H^2(M, \mathcal{O}) \oplus H^1(M, T^{1,0}) \oplus H^0(M, \wedge^2 T^{(1,0)})$, find $\Gamma \in C^\infty\Big(M, \wedge^2(T^{*(0,1)} \oplus T^{(1,0)})\Big)$ such that

$$\Gamma_1 \equiv_1 \Gamma, \quad \overline{\partial}\Gamma + \frac{1}{2}[\![\Gamma, \Gamma]\!] = 0.$$

5. Use LWX-theory for Γ to get $\overline{L}_\Gamma$.
6. Use LWX-theory for $\overline{\Gamma}$ to get $L_{\overline{\Gamma}}$.
7. $(L_{\overline{\Gamma}}, \overline{L}_\Gamma)$ is a new generalized complex structure.
8. Sometimes, the deformed object could be a symplectic structure.

Issue and motivation

Theorem (Moser, 65)

Symplectic structures on compact manifolds are rigid.

Gautieri 04, Poon 06 (Kodaira-Thurston surfaces)

Theorem (Gray, 59)

Contact structures on compact manifolds are rigid.

Problem

Is it possible to enlarge the category of geometry so that contact structures could be deformation in a non-trivial and controlled manner?

Remarks:

- Similarity
- Difference
- Classical structures: Jacobi, Dirac, conformal Dirac, Lichnerowicz.

Generalized almost contact structures on M^{2n+1}

Definition (After Vaisman 07)

A generalized almost contact structure is a collection of tensors: $\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$, $\xi + \eta \in C^\infty(M, TM \oplus T^*M)$

$$\Phi = \begin{pmatrix} \varphi & \pi \\ \theta & -\varphi^* \end{pmatrix} : TM \oplus T^*M \to TM \oplus T^*M$$

such that $\Phi + \Phi^* = 0$, $\eta(\xi) = 1$, $\Phi(\xi) = 0$, $\Phi(\eta) = 0$, $\Phi \circ \Phi = -\mathbb{I} + \xi \odot \eta$.
where $(\xi \odot \eta)(X + \alpha) := \eta(X)\xi + \alpha(\xi)\eta$.

$$\Phi_{\text{ker}} : \ker \eta \oplus \ker \xi \to \ker \eta \oplus \ker \xi, \quad \Phi_{\text{ker}} \circ \Phi_{\text{ker}} = -\mathbb{I}.$$

Remark: Focus on tensorial objects only. No equivalence. Formal.

Subbundles

$\Phi_{\text{ker}}^2 = -\mathbb{I}$.

$E^{1,0} = \{e - i\Phi(e) : e \in C^\infty(M, \ker\eta \oplus \ker\xi)\}$

$$L := L_\xi \oplus E^{1,0}, \quad \overline{L} = L_\xi \oplus E^{0,1}, \quad L^* = L_\eta \oplus E^{0,1}, \quad \overline{L}^* = L_\eta \oplus E^{1,0}.$$

$$L \oplus L^* = (TM \oplus T^*M)_{\mathbb{C}}, \quad \overline{L} \oplus \overline{L}^* = (TM \oplus T^*M)_{\mathbb{C}}.$$

Fact: $E^{1,0}$, $E^{0,1}$, $L, \overline{L}, L^*, \overline{L}^*$ are isotropic.
But

$$\overline{L} \neq L^* \qquad !!$$

Integrability, or the lack of it

Definition

Given a $\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$-structure, if $C^\infty(M, L)$ is Courant-closed, (but $C^\infty(M, L^*)$ is not necessarily closed,) then $\mathcal{J}$ is a generalized contact structure.

Remember: $\overline{L} \neq L^*$!!

Definition

Given a $\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$-structure, if both $C^\infty(M, L)$ and $C^\infty(M, L^*)$ are Courant-closed, then $\mathcal{J}$ is a generalized "complex" structure. (Even though $\dim M = 2n + 1$.)

Key: (Not) Lie bialgebroid.

Avoiding terminology: "generalized normal contact structures".

Obstruction

Problem

Assume $C^\infty(M, L)$ *is closed, determine whether or when* $C^\infty(M, L^*)$ *is also closed.*

LWX's obstruction for formation of Lie bialgebroids: For any three sections v_0, v_1, v_2 of $L^* = L_\eta \oplus E^{0,1}$,

$$\mathrm{Nij}(v_0, v_1, v_2) = \frac{1}{3}(\langle [\![v_0, v_1]\!], v_2\rangle + \langle [\![v_1, v_2]\!], v_0\rangle + \langle [\![v_2, v_0]\!], v_1\rangle).$$

$\mathrm{Nij} \in C^\infty(M, \wedge^3 L), L = L_\xi \oplus E^{1,0}, \wedge^3 L = \wedge^3 E^{1,0} \oplus L_\xi \otimes \wedge^2 E^{1,0}$.

Proposition

Given $\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$ *and* $C^\infty(M, L)$ *closed. Then* L^* *is closed if and only if* $\xi \wedge (\rho^* d\eta)^{2,0} = 0$*, where* $\rho : E^{1,0} \to TM_{\mathbb{C}}$.

Odd dimensional analogue of symplectic structures

Definition (Libermann, 1958)

An almost cosymplectic structure on M^{2n+1} is a reduction from $\mathrm{GL}(2n+1,\mathbb{R})$ to $\mathrm{Sp}(n,\mathbb{R})$. That is the choice of a 1-form η and a 2-form θ such that $\eta \wedge \theta^n \neq 0$ everywhere.

Definition

(η, θ) is a cosymplectic structure if $d\eta = 0$ and $d\theta = 0$.

Definition

η is a contact 1-form on M^{2n+1} if $\eta \wedge (d\eta)^n \neq 0$ everywhere.

A contact 1-form determines an almost cosymplectic structure $(\eta, d\eta)$, but it is NEVER a cosymplectic structure without qualification.

As generalized almost contact structures

- Almost cosymplectic (η, θ): (1-form, 2-form). $\eta \wedge \theta^n \neq 0$ everywhere. Define $\flat : TM \to T^*M$ by

$$\flat(X) = \iota_X \theta - \eta(X)\eta, \qquad \text{then } \pi(\alpha, \beta) := \theta(\flat^{-1}(\alpha), \flat^{-1}(\beta)).$$

  $\flat$ is an isomorphism. There exists a unique ξ such that $\eta(\xi) = 1$ and $\iota_\xi \theta = 0$. Choose $\varphi = 0$.

$$\Phi = \begin{pmatrix} 0 & \pi \\ \theta & 0 \end{pmatrix}.$$

- If η is a contact 1-form, choose $\theta = d\eta$. Then follow the above construction.

Odd dimensional analogue of complex structures

Definition (Sasaki (60))

A (ξ, η, φ)-structure on M^{2n+1} consists of φ a (1,1)-tensor, a vector field ξ and a 1-form η such that $\varphi^2 = -\mathbb{I} + \eta \otimes \xi$, and $\eta(\xi) = 1$.

Definition

A (ξ, η, φ)-structure on M is "normal" if and only if a naturally defined almost complex structure on $M \times \mathbb{R}^+$ is integrable. Equivalently, $\mathcal{L}_\xi \varphi = 0$, $\mathcal{L}_\xi \eta = 0$, and $\mathcal{N}_\varphi = -\xi \otimes d\eta$, where
$\mathcal{N}_\varphi(X, Y) := [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi([\varphi X, Y] + [X, \varphi Y])$.

A contact 1-form η does not determine a (ξ, η, φ)-structure until a "compatible" metric is chosen. φ is metric dependence. Too many choices.
$\mathcal{J} = (\xi, \eta, \varphi, \pi = 0, \theta = 0)$.

Integrability of classical examples

Theorem (Examples of generalized contact structures)

$C^\infty(M, L)$ is closed for

- *Cosymplectic (η, θ). i.e. $d\eta = 0$ and $d\theta = 0$.*
- *Contact 1-form η. i.e. $\theta = d\eta \neq 0$.*
- *Normal (ξ, η, φ)-structures. i.e. $\mathcal{N}_\varphi = -\xi \otimes d\eta$.*

Proof: DBH.

Theorem (Examples of generalized complex structures)

Both $C^\infty(M, L)$ and $C^\infty(M, L^*)$ are closed for

- *Cosymplectic (η, θ). (G-structure)*
- *Normal (ξ, η, φ)-structure. (Sasaki Cone)*

Proof: For the latter, check the "type" of $d\eta$.

Focus on contact 1-form η

Local picture: (x_j, y_j, z) on $\mathbb{R}^{2n+1}$.

$$\eta = dz - \sum_j y_j dx_j. \quad \xi = \frac{\partial}{\partial z}, \quad \theta = d\eta = \sum_j dx_j \wedge dy_j.$$

$$X_j := \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}, \quad Y_j = \frac{\partial}{\partial y_j}, \quad \pi = \sum_j X_j \wedge Y_j.$$

The obstruction $(\rho^* d\eta)^{2,0}$ is equal to

$$\frac{1}{4} \sum_j (dx_j - iY_j) \wedge (dy_j + iX_j).$$

$(\rho^* d\eta)^{2,0} + (\rho^* d\eta)^{0,2} = \frac{1}{4}(d\eta - \pi).$

Proposition

The obstruction for L^* being closed is not equal to zero anywhere when $\mathcal{J}$ is due to a contact 1-form.

New Examples

For generalized complex structures (on odd-dimensional manifolds), use LWX's Lie bialgebroid theory.
Deformation of classical cosymplectic structures away from classical objects e.g. H_3 Heisenberg group or cocompact quotient. Co-symplectic structure.
For generalized contact (not complex)?
Deformation theory due to Lie bialgebroid structure fails.
Alternative:

Proposition

Let M be the principal $SO(2)$-bundle over N with connection η and curvature $d\eta = p^*\omega$. Then the family J_t of generalized complex structures on N is lifted to a family $\mathcal{J}_t$ of generalized contact structures on M.

Proof. A Boothby+Wang type theorem. In their 1958 paper: *"On contact manifolds"*. (A backbone)

More new examples

On N the Kodaira surface, there exists

- a complex structure $J = J_0$,
- a symplectic form $\omega = J_1$, with ω being type (2,0)+(0,2) w.r.t. J.
- a family of generalized complex structures J_t containing J_0 and J_1.

Use Boothby-Wang construction.
Get a family of generalized contact structures $\mathcal{J}_t$. $\mathcal{J}_1$ is contact. $\mathcal{J}_t$ are non-classical objects for $t \neq 1$.

Remark: No more "Gray's Theorem" :-)

Remark: No deformation theory :-(

Further development

Contact 1-form and Reeb field: $\iota_\xi \eta = 0$. $\mathcal{L}_\xi \eta = 0$.

Theorem

$\mathcal{J} = (\xi, \eta, \pi, \theta, \varphi)$ a generalized contact structure (not necessarily cx), with $\mathcal{L}_\xi \eta = 0$, then

- *$\mathcal{L}_\xi \mathcal{J} = 0$; and*
- *the pair $E^{1,0}$ and $E^{0,1}$ forms a transversal Lie bialgebroid over (M, ξ).*

Reversing Boothby-Wang construction.
Cohomology theory.
Deformation.
Equivalence.
Another story.

Conclusion: contact vs symplectic.
Non-integrability vs integrability. Difference vs similarity.