Einstein Metrics,

Complex Surfaces,

and

Symplectic 4-Manifolds

Claude LeBrun
Stony Brook University

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But this talk concerns the case of dimension $n = 4$, where Kris also proved a number of interesting results.

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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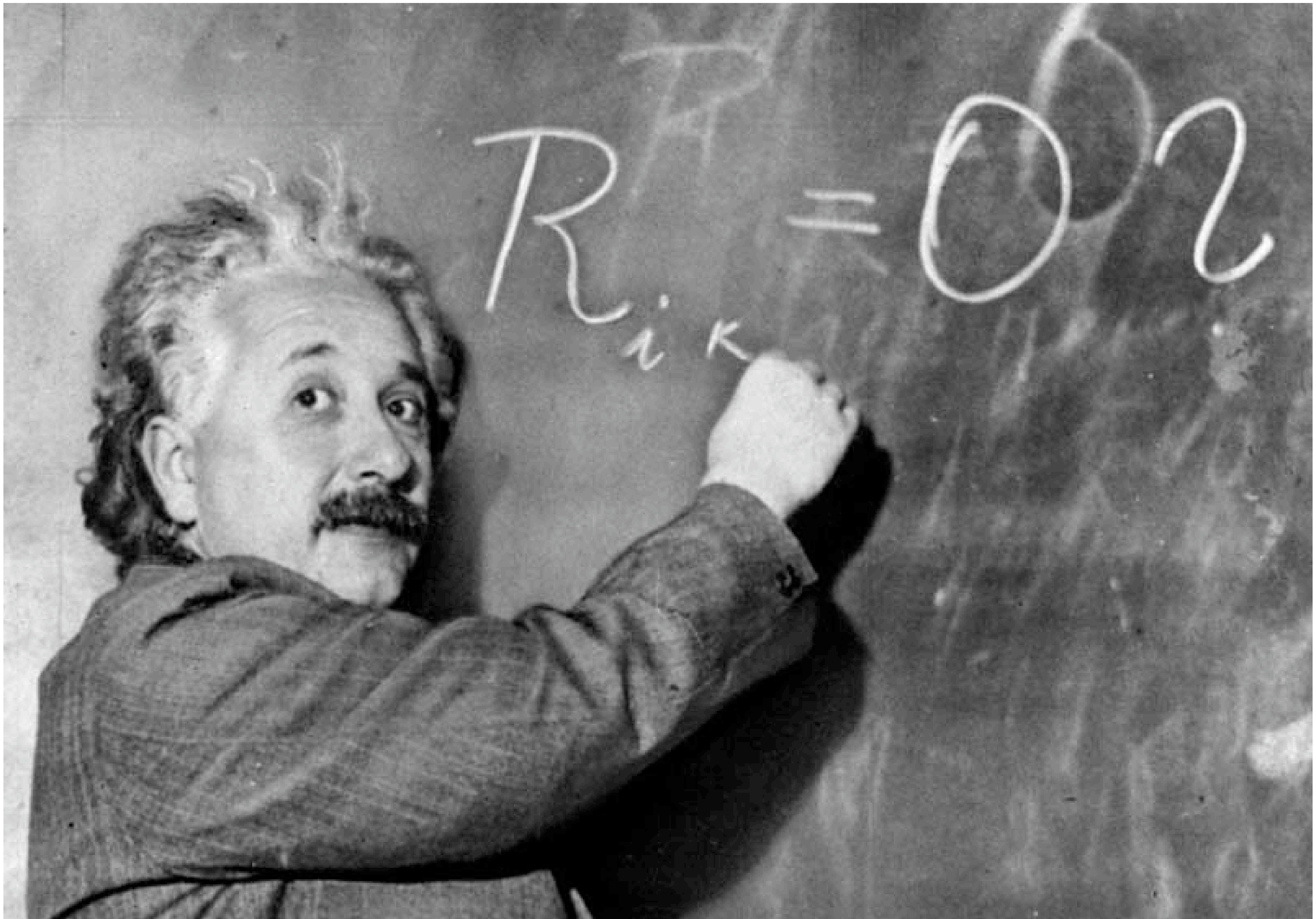

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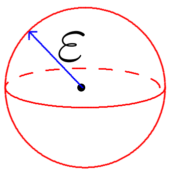
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

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where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature

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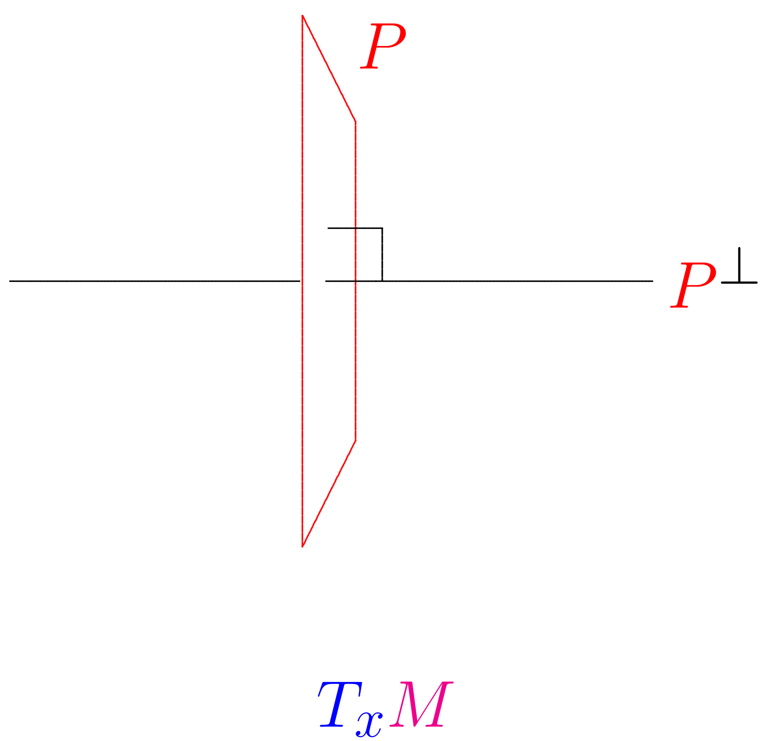

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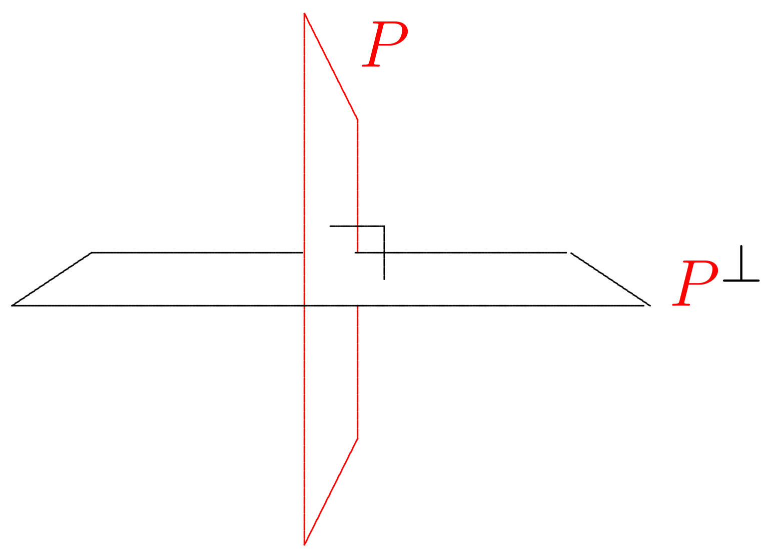

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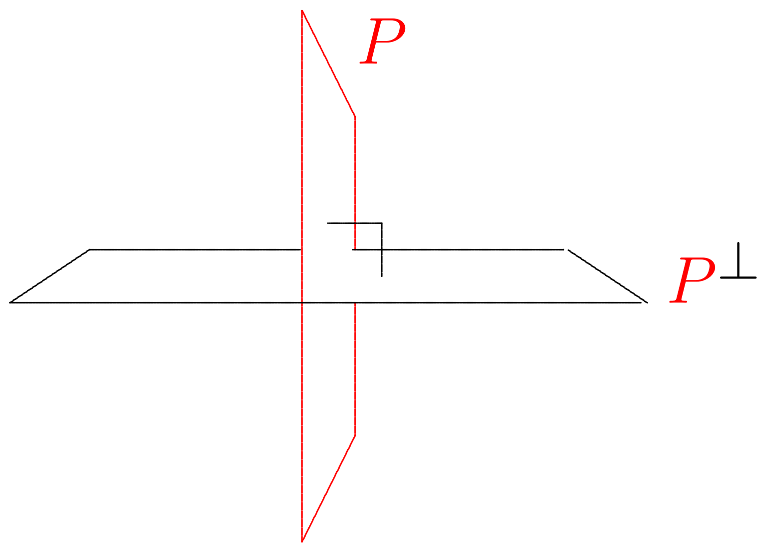

$$K(P) = K(P^{\perp})$$

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Symplectic Analog. *If* M^4 *is a smooth compact 4-manifold admits a symplectic form* ω*, when does* M^4 *also admit Einstein metrics?*

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(Warning: In rare circumstances,
h could still be Kähler for some $\tilde{J} \neq J$!)

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In $\lambda < 0$ case, corresponding questions still open.
Will try to briefly indicate what's currently known.

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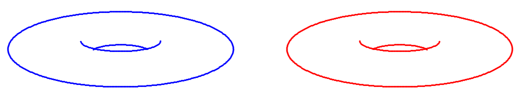

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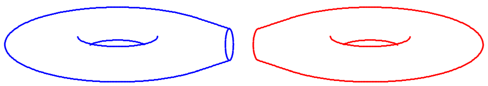

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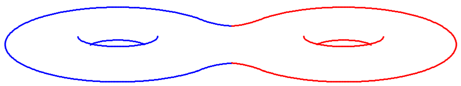

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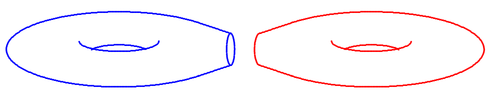

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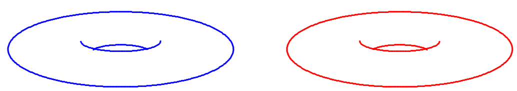

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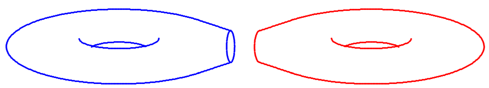

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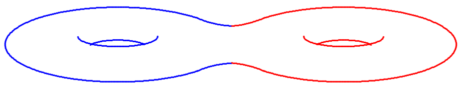

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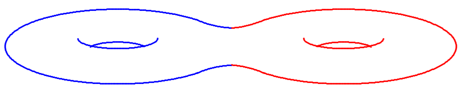

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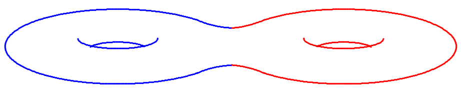

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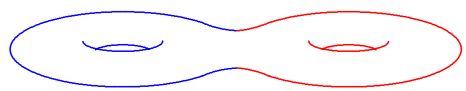

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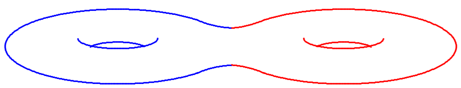

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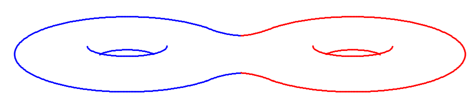

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in which new $\mathbb{CP}_1$ has self-intersection -1.

Theorem. *Suppose that* M *is a smooth compact oriented* 4*-manifold which admits a complex structure* J. *Then* M *also admits an (unrelated) Einstein metric* g *with* $\lambda > 0$

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

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Diffeotypes: Del Pezzo surfaces. ($\exists J$ with $c_1 > 0$.)

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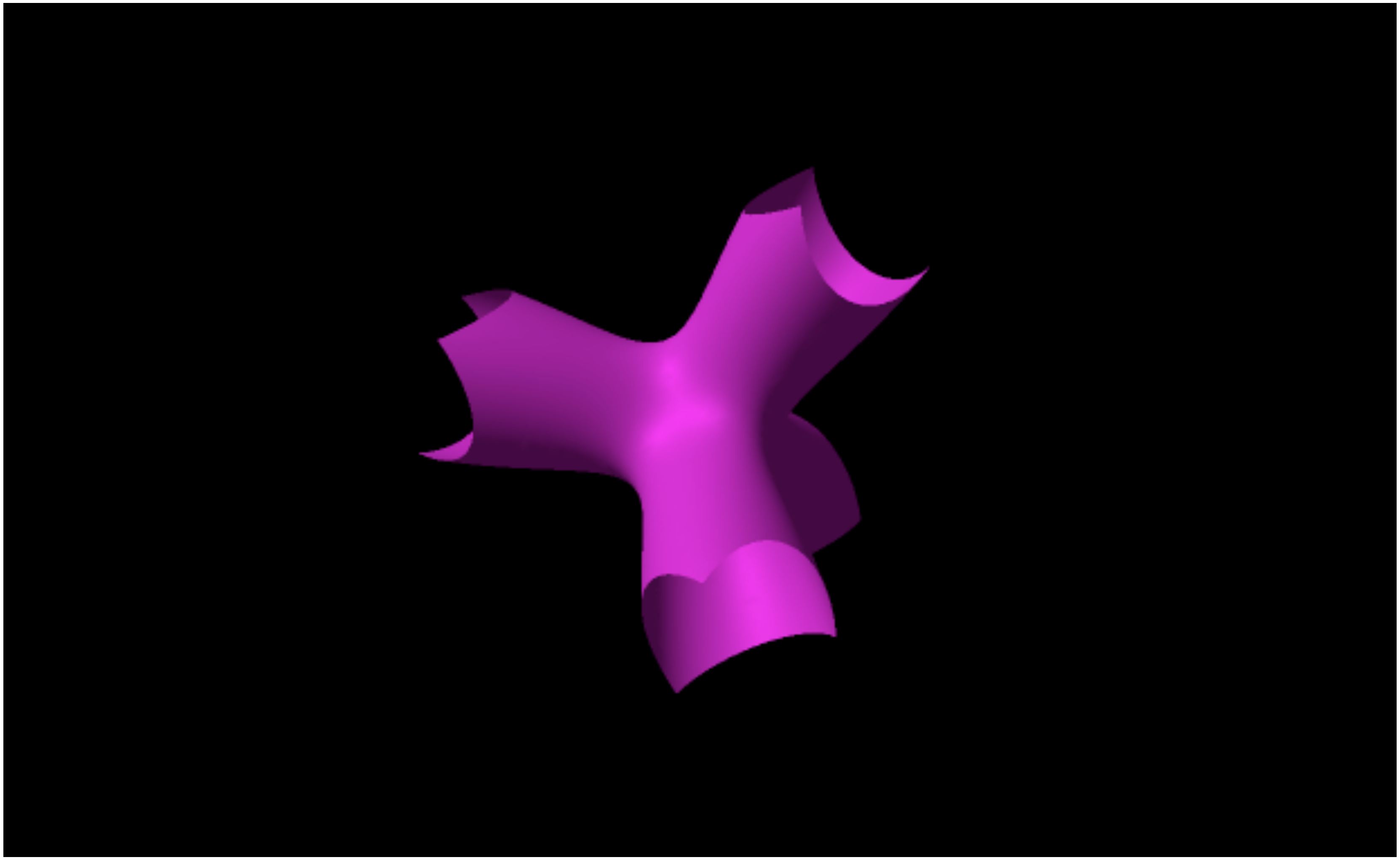

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Differentiable model for relevant $\mathbb{Z}_2$-action:

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In cases other than Del Pezzo surfaces:
also know moduli space of all Einstein metrics.

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We begin with existence.

Einstein metrics which are Kähler

Kähler-Einstein metrics

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Of course, $\mathbb{CP}_2$ and $S^2 \times S^2$ also admit K-E metrics with $\lambda > 0$ — namely, obvious homogeneous ones!

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Einstein metric is $g = s^{-2}h$.

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so rescaling $g \rightsquigarrow s^{-2} g$ gives metric with

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$$\frac{d}{dt}\mathcal{W}(g_t)\Big|_{t=0} = \int \dot{g}^{ab} B_{ab} \, d\mu_g$$

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Allows one to locate target Kähler class ($\neq c_1$!)

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Limit complex structure: toric geometry.

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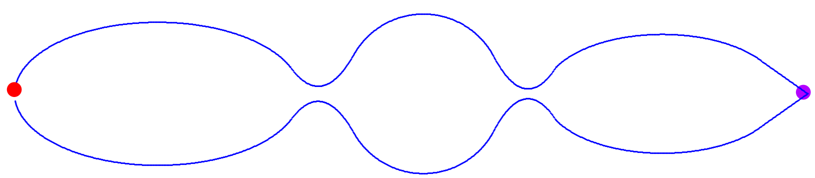
"Bubbling"



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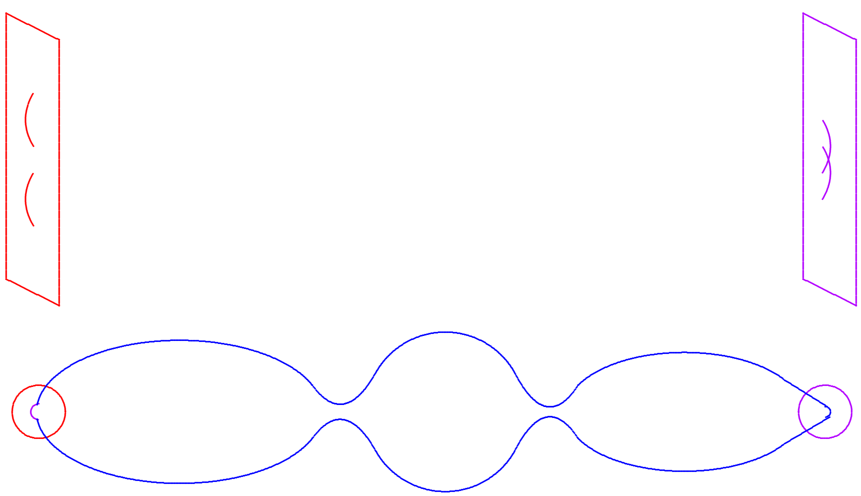

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Argument uses twistor theory, toric geometry.

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Conjecture. *Let (M^4, J) be a compact complex surface, and suppose that h is an Einstein metric on M which is Hermitian with respect to J:*

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Proof is work in progress.

Proposition (L '96). *Let* (M^4, J) *be a compact complex surface, and suppose that* h *is an Einstein metric on* M *which is Hermitian with respect to* J*:*

$$h(J\cdot, J\cdot) = h.$$

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- *after normalization,* $h = s^{-2} g$*.*

Theorem. *Suppose that M is a smooth compact oriented 4-manifold which admits either a complex structure or a symplectic structure. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \textit{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

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Will now discuss obstructions to Einstein metrics.

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$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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Theorem (Hitchin-Thorpe Inequality). *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if (M, g) finitely covered by flat T^4 or Calabi-Yau $K3$.

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In particular, in the complex case, (M, J) is either rational or of general type.

In the ${c_1}^2(M) > 0$ case, there is then a well-defined Seiberg-Witten invariant of M, for the spin^c structure induced by J or ω.

Seiberg-Witten theory:

generalized Kähler geometry of non-Kähler 4-manifolds.

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But $\bar{\partial} + \bar{\partial}^*$ does generalize:

spin^c Dirac operator, preferred connection on L.

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Let $L = \Lambda^{0,2}$ be its anti-canonical line bundle.

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Every unitary connection A on L induces spin^c Dirac operator

$$D_A : \Gamma(\mathbb{V}_+) \to \Gamma(\mathbb{V}_-)$$

generalizing $\bar\partial + \bar\partial^*$.

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$$D_A \Phi = 0$$

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Non-linear, but elliptic once 'gauge-fixing'

$$d^*(A - A_0) = 0$$

imposed to eliminate automorphisms of $L \to M$.

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If, in addition, ${c_1}^2 > 0$,
$\Longrightarrow \nexists g$ with $s \geq 0$.

Complex case:

Del Pezzo by Enriques and Kodaira.

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Theorem. *Suppose that M is a smooth compact oriented 4-manifold which admits either a complex structure or a symplectic structure. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if M is diffeomorphic to*

- *a Del Pezzo surface,*
- *a K3 surface,*
- *an Enriques surface,*
- *an Abelian surface, or*
- *a hyper-elliptic surface.*

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Theorem (Aubin/Yau). *Compact complex manifold* (M^{2m}, J) *admits compatible Kähler-Einstein metric with* $\lambda < 0 \iff c_1(M, J) < 0$.

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When $m = 2$, such M are necessarily minimal complex surfaces of general type.

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A complex surface M is of general type $\Longleftrightarrow$ its minimal model X satisfies

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(Better than Hitchin-Thorpe by a factor of 3.)

So being "very" non-minimal is an obstruction.

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Same conclusion holds in symplectic case.

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If so, quite different from Kähler-Einstein metrics!