G2 MANIFOLDS AND INTEGRABLE EQUATIONS

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David Baraglia, *Moduli of coassociative submanifolds and semi-flat coassociative fibrations,* **arXiv:0902.2135**

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J. Loftin, S-T. Yau & E. Zaslow, *Affine manifolds, SYZ geometry and the Y vertex*, J. Differential Geom. **71** (2005) 129–158.

CALABI-YAU MANIFOLDS

- Kähler form ω, holomorphic n-form $\Omega = \Omega_1 + i\Omega_2$

- L^n special Lagrangian submanifold: $\omega|_L = 0$, $\Omega_1|_L = 0$

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- L^n special Lagrangian submanifold: $\omega|_L = 0$, $\Omega_1|_L = 0$

- normal X: $d(i_X\omega) = 0$, $d(i_X\Omega_1) = 0$ and $i_X\Omega_1|_L = *i_X\omega|_L$

- moduli space M of deformations of L, $T_LM \cong H^1(L, \mathbf{R})$

R C McLean, *Deformations of calibrated submanifolds,* Comm. Anal. Geom. **6** (1998) 705–747.

- $\theta \in \Omega^1(M, H^1(L, \mathbf{R}))$

- $d\theta = 0$ (flat connection on bundle $H^1(L_m, \mathbf{R})$, $m \in M$)

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- $\Rightarrow$ local diffeomorphism $M \cong H^1(L, \mathbf{R})$

- flat affine structure on M

- $\theta(X) = [i_X\omega] \in H^1(L, \mathbf{R})$

- $\theta'(X) = [i_X\Omega_1] \in H^{n-1}(L, \mathbf{R})$

- $\Rightarrow$ local diffeomorphism $M \cong H^{n-1}(L, \mathbf{R})$

- $M \subset H^1(L, \mathbf{R}) \times H^{n-1}(L, \mathbf{R})$

NJH, *The moduli space of special Lagrangian submanifolds,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1997) 503 – 515

- $H^1(L, \mathbf{R}) \times H^{n-1}(L, \mathbf{R}) = V \times V^*$

- pairing $\Rightarrow$ symplectic form

- $M \subset H^1(L, \mathbf{R}) \times H^{n-1}(L, \mathbf{R})$ is Lagrangian

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- pairing $\Rightarrow$ symmetric form (signature (b_1, b_1)).

- induced positive definite metric on M

- $V \times V^* = T^*V$

- Lagrangian $\Rightarrow$ graph of $d\phi$

- metric

$$g = \sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j$$

G2 MANIFOLDS

- three-form φ, $d\varphi = 0$, $d\psi = d * \varphi = 0$

- L^4 coassociative submanifold: $\varphi|_L = 0$.

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- L^4 coassociative submanifold: $\varphi|_L = 0$.

- normal X: $d(i_X\varphi) = 0$, $i_X\varphi|_L$ self-dual 2-form

- moduli space M of deformations of L, $T_L M \cong H^2_+(L, \mathbf{R})$

- intersection form on $H^2(L, \mathbf{R})$, signature (p, q)

- locally $M \subset H^2(L, \mathbf{R})$

- induced positive definite metric on M (cup product positive on $H^2_+(L, \mathbf{R})$)

EXAMPLE

- Z Calabi-Yau threefold $\Rightarrow S^1 \times Z$ is a G_2-manifold

- L special Lagrangian, $S^1 \times L$ coassociative

- $H^2(S^1 \times L) \cong H^1(L, \mathbf{R}) \oplus H^2(L, \mathbf{R})$

SEMI-FLAT CALABI-YAU MANIFOLDS

- Z Calabi-Yau n-manifold , $L^n \subset Z$ torus

- $\dim M = H^1(L, \mathbf{R}) = n$

- if normal vectors non-vanishing, Z fibres over M

- Lagrangian torus fibration

Defn: Z is semiflat if there is a T^n action, preserving ω and Ω, such that the orbits are special Lagrangian.

- $T^n \to Z \to M$
- homogeneous metric on fibres
- volume of fibres is a constant c^2 ($\Omega_2|_L$ is the volume form, and $d\Omega_2 = 0$)

THE HORIZONTAL DISTRIBUTION

- $TL^{\perp} = I(TL)$ defines the horizontal distribution.

- $0 = [X, Y] + I[IX, Y] + I[X, IY] - [IX, IY]$ (Nijenhuis tensor)

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- $= \mathcal{L}_Y(X) - \mathcal{L}_X(Y) - [IX, IY] = -2[X, Y] - [IX, IY]$ (action preserves I)
- $= -[IX, IY]$ (abelian group action)

THE QUOTIENT METRIC

- X vector field from T^n action, IX normal vector field

- $i_{IX}\omega = X^\flat$

$$\int_L X^\flat \wedge *X^\flat = (X,X)\,\mathrm{vol}(L)$$

quotient metric:

$$c^{-2}\sum_{i,j}\frac{\partial^2\phi}{\partial x_i \partial x_j}dx_i dx_j$$

THE CALABI-YAU METRIC

$$g_{ij} = c^{-2}\frac{\partial^2\phi}{\partial x_i \partial x_j}dx_i dx_j + \frac{\partial^2\phi}{\partial x_i \partial x_j}dy_i dy_j$$

- $z_i = x_i + icy_i$, Kähler form $\partial\bar{\partial}\phi$
- Ricci-flat iff

$$\det \frac{\partial^2\phi}{\partial x_i \partial x_j} = const.$$

SEMI-FLAT* G2 MANIFOLDS

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$$\frac{4}{7} = .5714^{*}....$$

Defn: A G_2 manifold Z is semiflat if there is a T^4 action, preserving φ, such that the orbits are coassociative.

- homogeneous metric on fibres
- volume of fibres is a constant c^2 ($\psi|_L$ volume form and $d\psi = 0$).
- $\dim H^2_+(L, \mathbf{R}) = 3$
- $T^4 \to Z \to M^3$

THE HORIZONTAL DISTRIBUTION

- $T \cong \Lambda^1 \oplus \Lambda^2_+$ as $SO(4)$ representations

- $\psi = e_1 \wedge e_2 \wedge e_3 \wedge e_4 + (e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_6 \wedge e_7 + (e_1 \wedge e_3 + e_4 \wedge e_2) \wedge e_7 \wedge e_5 + (e_1 \wedge e_4 + e_2 \wedge e_3) \wedge e_5 \wedge e_6$

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$$\bigwedge^m (\Lambda^1 \oplus \Lambda^2_+) = \bigoplus_{p+q=m} \Lambda^p \otimes \Lambda^q(\Lambda^2_+)$$

- X horizontal iff $i_X\psi$ is of type $(2,1)$

- for invariant forms, α type $(p, q) \Rightarrow d\alpha$ of type $(p, q+1)$

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- $\mathcal{L}_X i_Y \psi = i_{[X,Y]}\psi + i_Y d(i_X\psi)$ $(d\psi = 0)$

- $i_{[X,Y]}\psi = d(i_X i_Y \psi) + i_X d(i_Y\psi) - i_Y d(i_X\psi)$ type $(2,1)$

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associative foliation

THE THREE-FORM

- $u : M \to H^2(L) =$ invariant 2-forms on T^4

$$\varphi = c\,d\,\mathrm{vol}_M + du = c\,d\,\mathrm{vol}_M + \sum_i dx_i \wedge \frac{\partial u}{\partial x_i}$$

- $d\psi = 0 = d * \varphi = d * u$: u is harmonic

- $u : M \to H^2(L) = \mathbf{R}^{3,3}$

- u harmonic

- induced metric

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$\Rightarrow$ minimal submanifold

CALABI-YAU?

- $u(M) \subset V \times V^*$: Monge-Ampère equation

- $\Omega_1 + i\Omega_2 = dz_1 \wedge dz_2 \wedge dz_3$ decomposable, similarly $\Omega_1 - i\Omega_2$

CALABI-YAU?

- $u(M) \subset V \times V^*$: Monge-Ampère equation

- $\Omega_1 + i\Omega_2 = dz_1 \wedge dz_2 \wedge dz_3$ decomposable, similarly $\Omega_1 - i\Omega_2$

- $v_1 \in \Lambda^3 V$, $v_2 \in \Lambda^3 V^*$ decomposable

- Monge-Ampère $\Rightarrow$ $v_1 - v_2$ vanishes on $u(M)$: "special Lagrangian" $\Rightarrow$ minimal Lagrangian

THE GAUSS MAP

- E A Ruh and J Vilms, *The tension field of the Gauss map,* TAMS **149** (1970) 569 – 573

- The Gauss map of a minimal submanifold of $\mathbf{R}^n$ is harmonic.

- $g : M \to SO(3,3)/S(O(3) \times O(3)) = SL(4, \mathbf{R})/SO(4)$

- K Corlette, *Flat G-bundles with canonical metrics,* J. Differential Geom. **28** (1988) 361– 382

TWO-DIMENSIONAL REDUCTIONS

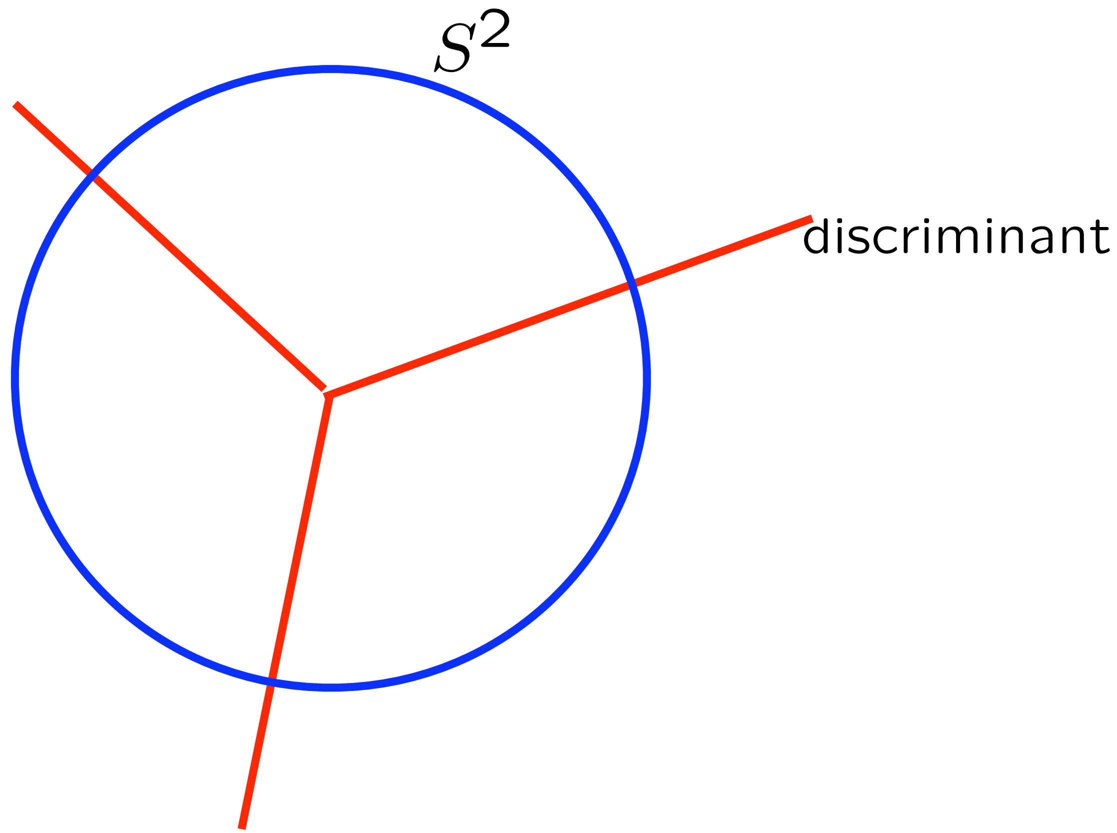
THE Y-VERTEX

- J. Loftin, S-T. Yau & E. Zaslow *Affine manifolds, SYZ geometry and the Y vertex*, J. Differential Geom. **71** (2005) 129–158.

- $T^3 \to Z \to M$

- discriminant locus in M codimension 2

- H Ooguri & C Vafa *Summing up Dirichlet instantons*, Phys. Rev. Lett. **77** (1996) 3296–3298.


S^2
discriminant

- Radially symmetric solution of Monge-Ampère

- metric $e^f|dz|^2$ on $\mathbf{C}P^1$

$$f_{z\overline{z}} + |U|^2 e^{-2f} + \frac{1}{2}e^f = 0$$

- Udz^3 cubic differential form with double poles at the singu larities

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$$f_{z\bar{z}} + |U|^2 e^{-2f} + \frac{1}{2}e^f = 0 \qquad \text{Tzitzeica equation}$$

- Udz^3 cubic differential form with double poles at the singu larities

THE G2 CASE

- (Z, φ) G_2 semi-flat manifold
- U vector field commuting with T^4 action
- $\varphi = c d\,\mathrm{vol}_M + du$
- $\mathcal{L}_U du = \lambda du \Rightarrow$ minimal 3-manifold is a cone $dr^2 + r^2 g_\Sigma$

Case 1: $\lambda = 0$

$M = \mathbf{R} \times \mathbf{\Sigma} \subset \mathbf{R} \times \mathbf{R}^{2,3} \Rightarrow$

minimal surface $\mathbf{\Sigma}$ in $\mathbf{R}^{2,3}$ (Weierstrass representation)

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minimal surface $\mathbf{\Sigma}$ in $\mathbf{R}^{2,3}$ (Weierstrass representation)

Case 2: $\lambda \neq 0$

$M \subset \mathbf{R}^{3,3}$ is a cone $dr^2 + r^2 g_{\mathbf{\Sigma}} \Rightarrow$

minimal surface $\mathbf{\Sigma}$ in quadric $SO(3,3)/SO(2,3)$.

$$\begin{aligned} f_{z\bar{z}} &= -e^{g-f} - e^{f} \\ g_{z\bar{z}} &= |U|^2 e^{-g} + e^{g-f} \end{aligned}$$

Put $e^{2g} = |U|^2 \Rightarrow$ Tzitzeica equation

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Affine Toda equations

Put $e^{2g} = |U|^2 \Rightarrow$ Tzitzeica equation

J Bolton, F Pedit & L M Woodward , *Minimal surfaces and the affine Toda field model,* J. reine angew. Math. **459** (1995) 119–150.

NEW DIRECTIONS?

- $H^1(L_m, \mathbf{R})$ flat vector bundle over M^3

- metric $\Rightarrow$ reduction of structure from $SL(4, \mathbf{R})$ to $SO(4)$

- Other structures: $Sp(4, \mathbf{R}), SO(3,1), SL(2, \mathbf{C}), SO(2,2)$....

- $H^1(L_m, \mathbf{R})$ flat vector bundle over M^3
- metric $\Rightarrow$ reduction of structure from $SL(4, \mathbf{R})$ to $SO(4)$
- Other structures: $Sp(4, \mathbf{R})$, $SO(3, 1)$, $SL(2, \mathbf{C})$, $SO(2, 2)$....

- $Sp(4, \mathbf{R}) \sim SO(2,3)$

- symplectic form $[\omega] \in H^2(T^4, \mathbf{R})$

- ω orthogonal to $u(M)$

- $u : M \to \mathbf{R}^{3,3}$

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- closed two-form ω
- $\omega^3 = 0$
- $\omega \wedge \psi = 0 \Rightarrow \omega \wedge \varphi = *\omega$: ω harmonic
- Global solutions?

WHY THE Y-VERTEX?

- No vertices $\Rightarrow$ Euler characteristic of $Z = 0$

- Restrictive for Calabi-Yau (M Gross)

- No vertices $\Rightarrow$ Euler characteristic of $Z = 0$
- Restrictive for Calabi-Yau (M Gross)
- Not an issue for G_2-manifolds
- Are there G_2 torus fibrations over knot complements in S^3?