Hilbert series and obstructions to asymptotic Chow semistability

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Kähler and Sasakian Geometry in Rome

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In memory of Krzysztof Galicki

1. Obstructions to Asymptotic Chow semistability (background)

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We seek a constant scalar curvature Kähler (cscK) metric in $c_1(L)$.

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Theorem A. (Donaldson, JDG 2001)
Let (M, L) be a polarized manifold with Aut(M, L) discrete.
If $\exists$ a cscK metric
then (M, L) is asymptotically Chow stable.

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Theorem (Ono-Sano-Yotsutani)

There are toric Fano Kähler-Einstein manifolds which are not asymptotically Chow-semistable (polystable).

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and this divisor is defined by

$\widehat{M}_k \in (\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$.

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Stabilizer of $\widehat{M}_k$ under $SL(V_k)$-action is $\mathrm{Aut}(M, L)$.

In Theorem A (Donaldson), "$\mathrm{Aut}(M, L)$ is discrete" means "the stabilizer is finite".

M is said to be Chow polystable w.r.t. L^k if the orbit of $\widehat{M}_k$ in $(\mathrm{Sym}^{d_k}(V_k))^{\otimes(m+1)}$ under the action of $\mathrm{SL}(V_k)$ is closed.

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M is asymptotically Chow polystable (resp. stable or semistable) w.r.t. L if there exists a $k_0 > 0$ such that M is Chow polystable (resp. stable or semistable) w.r.t. L^k for all $k \geq k_0$.

Theorem B (Mabuchi, Osaka J. Math. 2004)

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If Aut(M, L) is not discrete then

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Theorem C (Mabuchi, Invent. Math. 2005)

Let (M, L) be a polarized manifold, and suppose Aut(M, L) is not discrete.

If $\exists$ a cscK metric in $c_1(L)$ and if the obstruction vanishes then (M, L) is asymptotically Chow polystable.

$$\begin{aligned}
\mathfrak{h}_0 &= \{X \text{ holo vector field} \mid \text{zero}(X) \neq \emptyset\} \\
&= \{X \text{ holo vector field} \mid \exists u \in C^\infty(M) \otimes \mathbb{C} \\
&\qquad \text{s.t. } X = \text{grad}' u = g^{i\bar{j}} \frac{\partial u}{\partial \overline{z}^j} \frac{\partial}{\partial z^i}\} \\
&= \{X \text{ holo vector field} \mid \exists \text{lift to an infinitesimal action on } L\} \\
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Theorem D (F, Internat. J. Math. 2004)

Let (M, L) be a polarized manifold with $\dim_{\mathbb{C}} M = m$.

(1) The vanishing of Mabuchi's obstruction is equivalent to the vanishing of Lie algebra characters $\mathcal{F}_{\mathrm{Td}^i} : \mathfrak{h}_0 \to \mathbb{C}$, $i = 1, \cdots, m$.

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(2) $\mathcal{F}_{\mathrm{Td}^1}$ = obstruction to $\exists$ of cscK metric (Futaki invariant).

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Question (b) In Theorem D, if $\mathcal{F}_{\mathrm{Td}^1} = 0$ then $\mathcal{F}_{\mathrm{Td}^2} = \cdots = \mathcal{F}_{\mathrm{Td}^m} = 0$?

Question (c) $\dim \operatorname{span}\{\mathcal{F}_{\mathrm{Td}^1}, \cdots, \mathcal{F}_{\mathrm{Td}^m}\} = 1$?

Futaki-Ono-Sano, 2008

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Ono-Sano-Yotsutani recently showed the answers to Questions (a) and (b) are No.

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Then the lift $\widetilde{X}$ to L of X is written as

$$\widetilde{X} = \theta(\widetilde{X}) iz \frac{\partial}{\partial z} + X^\sharp.$$

The ambiguity of $\widetilde{X}$ is const $iz\frac{\partial}{\partial z}$.

$$
\begin{aligned}
\widetilde{X} + ciz\frac{\partial}{\partial z} &= \theta(\widetilde{X} + ciz\frac{\partial}{\partial z})iz\frac{\partial}{\partial z} + X^{\sharp} \\
&= (\theta(\widetilde{X}) + c)iz\frac{\partial}{\partial z} + X^{\sharp}.
\end{aligned}
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Given a Kähler form $\omega \in c_1(L)$, suppose the connection θ is so chosen that

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Conclusion: **Ambiguity of Hamiltonian function $\Longleftrightarrow$ ambiguity of lifting of X to L.**

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Put

$$L(X) = \nabla_X - L_X \in \Gamma(\mathrm{End}(T'M))$$

and let

$$\Theta \in \Gamma(\Omega^{1,1}(M) \otimes \mathrm{End}(T'M))$$

be the (1,1)-part of the curvature form of ∇.

Def: For $\phi \in I^p(GL(m,\mathbb{C}))$, we define

$$\begin{aligned}\mathcal{F}_\phi(X) \;=\; & (m-p+1)\int_M \phi(\Theta)\wedge u_X\,\omega^{m-p} \\ & + \int_M \phi(L(X)+\Theta)\wedge\omega^{m-p+1}.\end{aligned} \qquad (1)$$

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Theorem D : Vanishing of Mabuchi's obstruction is equivalent to

$$\mathcal{F}_{\mathrm{Td}^1}(X) = \cdots = \mathcal{F}_{\mathrm{Td}^{\mathrm{m}}}(X) = 0$$

for all $X \in \mathfrak{h}_0$.

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For $g \in T^{m+1}$,

$$L(g) := \sum_{k=0}^{\infty} \mathrm{Tr}(g|_{H^0(M, K_M^{-k})})$$

the formal sum of the Lefchetz numbers.

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Let $\{v_j \in \mathbb{Z}^m\}_j$ be the generators of 1-dimensional faces of the fan of M.

$\lambda_j := (v_j, 1) \in \mathbb{Z}^{m+1}$,
$C^* := \{y \in \mathbb{R}^{m+1} | \lambda_j \cdot y \geq 0, \forall j\} \subset \mathfrak{g}^* = (\mathrm{Lie}(T^{m+1}))^*$,
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C^* is a cone over P^*.

Integral points in $C^* \longleftrightarrow \cup_{k=1}^{\infty}$ basis of $H^0(M, K_M^{-k})$

For $\mathbf{x} \in T_{\mathbb{C}}^{m+1}$, we put

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Def: $\mathcal{C}(\mathbf{x}, C^*) := \sum_{\mathbf{a} \in C^* \cap \mathbb{Z}^{m+1}} \mathbf{x}^{\mathbf{a}}$ **Hilbert series**

Fact: $\mathcal{C}(\mathbf{x}, C^*)$ **is a rational function of** $\mathbf{x}$**.**

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Lemma : $\mathcal{C}(\mathbf{x}, C^*) = L(\mathbf{x})$**.**

For $\mathbf{b} \in \mathbb{R}^{m+1} \cong \mathfrak{g} = \mathrm{Lie}(T^{m+1})$**,**

$e^{-t\mathbf{b}} := (e^{-b_1 t}, \cdots, e^{-b_{m+1} t})$

$\mathcal{C}(e^{-t\mathbf{b}}, C^*) = \sum_{\mathbf{a} \in C^* \cap \mathbb{Z}^{m+1}} e^{-t\mathbf{a} \cdot \mathbf{b}}$ **: rational function in** t

$$C_R := \{(b_1, \cdots, b_m, m+1) | (b_1, \cdots, b_m) \in (m+1)P\} \subset \mathfrak{g}$$

where

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The tangent space of C_R at $(0, \cdots, 0, m+1)$ is

$T_{(0,\cdots,0,m+1)}C_R = \{\mathbf{c} = (c_1, \cdots, c_m, 0)\} \subset \mathfrak{g}$.

This defines another way of lifting of T^m-action to L.

Put $\mathbf{b} = (0, \cdots, 0, m+1)$.

Comparing the two liftings of T^m-action to L we can show

Theorem: (1) The coeffocients of the Laurant series of the rational function $\frac{d}{ds}|_{s=0}\mathcal{C}(e^{-t(\mathbf{b}+s\mathbf{c})}, C^*)$ in t span the linear space spanned by $\mathcal{F}_{\mathrm{Td}^1}, \cdots, \mathcal{F}_{\mathrm{Td}^m}$.

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(2) In dimension 2, the linear spans are 1-dimensional. In dimension 3 the linear spans are at most 2-dimensional, and there are examples in which the linear spans are 2-dimensional.

Remark Martelli-Sparks-Yau: From t^{-m} term we get the Futaki invariant.

Our computations show that the question is closely related to a question raised by Batyrev and Selivanova:

Our computations show that the question is closely related to a question raised by Batyrev and Selivanova:

Is a toric Fano manifold with vanishing $f (= \mathcal{F}_{\mathsf{Td}^1})$ for the anticanonical class necessarily symmetric?

Recall that a toric Fano manifold M is said to be symmetric if the trivial character is the only fixed point of the action of the Weyl group on the space of all algebraic characters of the maximal torus in $\mathrm{Aut}(M)$.

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Nill and Paffenholz gave a counterexample to Batyrev–Selyvanova.