Anti-self-dual bihermitian structures on surfaces of class VII

(joint work with M. Pontecorvo)

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Fujiki, A., and Pontecorvo, M.,
Anti-self-dual bihermitian structures
on Inoue surfaces,
arXive:math.DG/0903.1320v1, (2009).

Results: Existence of ASD-BH structures on hyperbolic and parabolic Inoue surfaces and their small deformations is shown by the twistor method of Donaldson-Friedman.

M compact oriented smooth 4-manifold

A <u>Bihermitian structure</u> (BH structure) is a triple $([g]; J_1, J_2)$, where

- $[g]$: a conformal structure
- $J_1,\ J_2$: complex structures

  such that

  a) J_i are compatible with oritentation

  b) g gives hermitian metrics on $S_i := (M, J_i),\ i = 1, 2$

  c) $J_1 \neq \pm J_2$

Anti-self-dual bihermitian structure is

$([g]; J_1, J_2)$ with $[g]$ anti-self-dual

i.e., $W_+ \equiv 0$ (self-dual Weyl curvature)

Example:

Hyperhermitian structure is:

a pair $(g, \{J_t\}_{t\in S^2})$, where

- J_t: complex structures on M
- g: a J_t-invariant metric

Take $J_1 := J_a, J_2 := J_b, a, b \in S^2, a \neq \pm b$;

then $([g], J_1, J_2)$ are ASD-BH

Necessary conditions

$([g], J_1, J_2)$ ASD-BH structure on M which is not hyperhermitian.

Then for $S = S_1, S_2$ we have:

- S is a surface of class VII

  (i.e. $b_1 = 1$ and $\kappa = -\infty$) (Boyer)

- S admits an effective and disconnected anti-canonical divisor (Pontecorvo)

Classification of surfaces as above with $\pi_1 \cong \boldsymbol{Z}$

(Nakamura, Pontecorvo, Dloussky, Apostlov-Grancharov-Gauduchon)

1) S minimal: S is one of the following:

- diagonal Hopf surface ($b_2 = 0$)
- parabolic Inoue surface ($b_2 \geq 1$)
- hyperbolic Inoue surface ($b_2 \geq 2$)

2) S not minimal: S is a blowing-up of one of the surfaces in 1) at a finite number of points lying on a fixed anti-canonical divisor

The above surfaces are all diffeo. to

$$M[m] := (S^1 \times S^3) \# m\bar{\boldsymbol{P}}^2$$

$\#$ connected sum

$\bar{\boldsymbol{P}}^2$ complex projective plane with ori. reversed

Anti-canonical divisors of surfaces in 1)

- diagonal Hopf surface

$$-K = E_1 + E_2$$

E_i smooth elliptic curves (in general unique)

- parabolic Inoue surface

$$-K = E + C$$

E a smooth elliptic curve,

C a cycle of rational curves (unique)

- hyperbolic Inoue surface

$$-K = C_1 + C_2$$

C_i cycles of rational curves (unique)

Known examples

- S diagonal Hopf surface

  $S = (\boldsymbol{C}^2 - 0)/\langle\mu\rangle, \mu : (z, w) \to (\alpha z, \beta w)$

  $$0 < |\alpha|, |\beta| < 1$$

  S carries ASD-BH $|\alpha| = |\beta|$

  (Pontecorvo)

- Certain blowing up diagonal Hopf surfaces and certain parabolic Inoue surfaces carries a ASD-BH structures (LeBrun)

Main Reustls:

S: hyperbolic Inoue surface with $b_2 = m \geq 2$

Theorem 1

$\exists$ real m-dimensional smooth family

$$\{([g]_t, J_{1,t}, J_{2,t})\}_{t \in U}$$

of ASD-BH structures on $M[m]$

with U a smooth manifold such that

$$S_{1,t} \cong S, \quad S_{2,t} \cong {}^{\mathbf{t}}S$$

where ${}^{\mathbf{t}}S$ is a transposition of S.

The family is universal at each t.

Theorem 2

$\exists$ real $3m$-dimensional smooth families

$$\{([g]_t, J_{1,t}, J_{2,t})\}_{t\in V}$$

of ASD-BH structures on $M[m]$

with V a smooth manifold,

extending the above family, such that:

if an 'anti-canonical pair' (S', C') is sufficiently close to the given (S, C), $S' \cong S_{1,t}$ for some $t \in V$.

The family is universal at each t.

Idea of Proof

S hyperbolic Inoue surface with

$$\overset{diffeo}{\sim} M := M[m]$$

Penrose correspondence

$$([g]_t, J_{1,t}, J_{2,t}) \Longleftrightarrow (Z_t, S^{\pm}_{1,t}, S^{\pm}_{2,t})$$

- Z_t : twistor space with smooth fibration

  $p : Z_t \to M$

- $L_x := p^{-1}(x), x \in M, \cong \boldsymbol{P}^1$: twistor lines

- $\sigma : Z \to Z$: real structure

- $S^{\pm}_{k,t} \subseteq Z_t$: smooth sections of p

  σ -conjugate to each other

Start of construction

1) smooth K-action on $m\bar{\boldsymbol{P}}^2$, $K := S^1 \times S^1$

2) $\exists$ an $(m-1)$-dim. smooth family of K-invariant ASD-structures on $m\bar{\boldsymbol{P}}^2$ (Joyce)

3) Z : associated twistor space with G-action, $G := \boldsymbol{C}^* \times \boldsymbol{C}^*$

Donaldson-Friedman method

1) L_i, L_j : twistor lines

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\mu} & Z \\ \bigcup & & \bigcup \\ Q_i \coprod Q_j & \to & L_i \coprod L_j \end{array}$$

blowing up with exceptional divisors

$Q_k := \mu^{-1}(L_k), k = i, j, \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$

2) $\varphi : Q_i \to Q_j$: suitable isomorphism

$\hat{Z} := \tilde{Z}/\varphi$: singular twistor space

with singular locus $Q \cong Q_k$

3) Z_t: smoothings of $\hat{Z}$ by complex analytic deformations; they are twistor spaces over $M[m]$ for 'real' t

Surfaces $S^{\pm}_{k,t}$

- (S^+_k, S^-_k), $1 \le k \le m+2$:
  σ-conjugate, G-invariant and $S^{\pm}_k \cdot L_x = 1$
- $S^{\pm}_k$ are smooth toric surfaces,
  intersect transversally along
  $L_k := S^+_k \cap S^-_k$: G-invariant twistor line
- Choose i and j with $|i-j| \ge 2$
- $\tilde{S}^{\pm}_l,\ \hat{S}^{\pm}_l,\ l = i, j$:
  proper transforms of $S^{\pm}_l$ in $\tilde{Z}$, $\hat{Z}$ ($\hat{S}^{\pm}_l$ are disjoint with ordinary double curve contained in Q)

Smoothing

$\hat{D} := (\hat{Z}, \hat{S}_i^{\pm}, S_j^{\pm}) \to D_t := (Z_t, S_{i,t}^{\pm}, S_{j,t}^{\pm})$:

a smoothing such that

$S_{i,t}^{\pm}$, $S_{j,t}^{\pm}$: hyperbolic Inoue surfaces which are transpositions of each other

<u>Structure of</u> $\hat{S}^{\pm}_{l,t}$

1) $C^{\pm}_i$: anti-canonical cycle of $S^{\pm}_i$

2) $L_i \subseteq C^{\pm}_i$ with $L_i^2 = 1$

3) $\tilde{S}^{\pm}_i \to S^{\pm}_i$ is a blow-up of $p^{\pm}_j \in C^{\pm}_i$

4) $\tilde{S}^{\pm}_i \to \hat{S}^{\pm}_i$ identification of L_i and E_i

5) $\hat{S}^{\pm}_i$ is thus a singular toric surface with ordinary double curve

6) $\exists$ smoothing $S^{\pm}_{l,t}$ of $\hat{S}^{\pm}_l$ which is a hyperbolic Inoue surface (Nakamura)

Important facts

- Any hyperbolic Inoue surface is obtained in this way
- By suitable choice of K-action and i, j any toric surface as above is produced
- Any smoothing $\hat{S}_l^{\pm} \to S_{l,t}^{\pm}$ extends to a smoothing $\hat{D} \to D_t$

<u>Smoothing result for</u> $(\hat{Z}, \hat{S})$

$(\hat{Z}, \hat{S})$: our main object, where

$\hat{S} := \hat{S}_i \bigcup \hat{S}_j, \hat{S}_l = \hat{S}_l^+ \bigcup \hat{S}_l^-$.

Kuranishi family of log-deformations of $(\hat{Z}, \hat{S})$

$$g : (\mathcal{Z}, \mathcal{S}) \to T,\ (Z_o, S_o) = (\hat{Z}, \hat{S}),\ o \in T$$

Theorem 3

1) T is smooth, $\dim T = 3m$,

$\exists D = \bigcup_{i=0}^{2m} D_i$ divisor with nc in T

such that

- if $t \in T - D_0$, Z_t is smooth and $S^{\pm}_{l,t}$ is a smooth surface of class VII with disconnected anti-canonical divisor
- $I := \bigcap_{i=1}^{2m} D_i$ (smooth, $\dim I = m$) if $t \in I - D_0$, $S^{\pm}_{l,t} \cong S, {}^{\mathbf{t}}S$, the given hyperbolic Inoue surface and its transposition (by a suitable choice of the initial data)

2) g is universal and σ induces a real structure on T canonically.

If $t \in T - D_0$ is a real point, $(Z_t, S^{\pm}_{i,t}, S^{\pm}_{j,t})$ is a desired twistor triple.

Computation of cohomological invariants

Theorem 4

1) (Obstructions)

$$H^2(\hat{Z}, \Theta_{\hat{Z}}(-\log \hat{S})) = 0$$

$$Ext^2_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) = 0$$

2) (Dimensions of moduli)

$$\dim H^1(\hat{Z}, \Theta_{\hat{Z}}(-\log \hat{S})) = m - 1$$

$$\dim Ext^1_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) = 3m$$

3) (Automorphism group)

$$\dim Ext^0_{O_{\hat{Z}}}(\Omega_{\hat{Z}}(\log \hat{S}), O_{\hat{Z}}) = 0$$

Problems:

Hyperbolic Inoue Case:

In $B_t := ([g]_t, J_{1,t}, J_{2,t})$ the m-dim. parameter is divided into $(m-1)$-dim. parameters for the initial Joyce metrics and 1-dim. smoothing parameters.

- Does the 'period' $\pm[\beta]_t \in H^1(M[m], \boldsymbol{R}) \cong \boldsymbol{R}$ goes to $\pm\infty$ when $t \to o$ along the smoothing parameter.
- What are the global moduli space of our structures like as a smooth orbifold ?
- A generalization of the construction: Start with a finite number of Joyce ASD structures and form a cycle of (blown-up) Joyce twistor spaces. Then try the same deformation construction. Are new ASD-BH structures obtained ?

- Relation of our ASD structures on $M[m]$ with those constructed by Joyce, which are invariant by local K-action.

  Both have the same m-dimensional parameters.

  Note that the universal covering of hyperbolic Inoue surface is a K-invariant domain of a toric surface.

  In our case the universal covering of $\hat{Z}$ admits a G-action.

Parabolic Inoue Case:

- Compare our ASD structures with those by LeBrun which are invariant by the semi-free S^1-action. In our case $(\hat{Z}, \hat{S})$ admits a $\boldsymbol{C}^*$-action. They seem to have the same number of parameters $m + 1$.
- Is the algebraic dimension of Z_t one for some t?