Einstein manifolds and triple systems

José Miguel Figueroa-O'Farrill

IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

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http://www.maths.ed.ac.uk/~jmf/Research/Talks/KSGR.pdf

Dipartimento di Matematica
"Guido Castelnuovo"
Università di Roma "La Sapienza"

in collaboration with
Dipartimento di Matematica
Università "Roma Tre"

will organize the

SECOND MEETING ON

"QUATERNIONIC STRUCTURES IN MATHEMATICS AND PHYSICS"

Roma, 6-10 September 1999

The Conference

After the first meeting held at SISSA in 1994 we would like to bring together scientists from different areas of Mathematics and Physics working in the field of quaternionic structures.
The conference will take place at Universita' di Roma "La Sapienza" and Universita' di Roma Tre, from September 6 to 10. The conference will start on Monday afternoon at Dipartimento di Matematica, Universita' La Sapienza . It is expected that during one day the conference will be hosted at Dipartimento di Matematica, Universita' di Roma Tre .

Speakers will include:

J. Armstrong (Oxford), L. Barberis (Cordoba), R. Bielawski (Edinburgh), O. Biquard (Palaiseau), E. Bonan (Paris), J.P.Bourguignon (IHES), V. Cortes (Bonn), I. Dotti-Miatello (Cordoba), J. Figueroa (Edinburgh), O. Hijazi (Nancy), D. Joyce (Oxford), D. Kaledin (Moscow), P. Kobak (Krakow), A. Moroianu (Palaiseau), Y. Nagatomo (Tsukuba), K. O'Grady (Roma), G. Papadopoulos (Cambridge), H. Pedersen (Odense), F. Podesta' (Firenze), Y. S. Poon (Riverside), M. Rocek (Stony Brook), J. Sawon (Oxford), U. Semmelman (Munchen), A. van Proeyen (Leuven), M. Verbitsky (Moscow).

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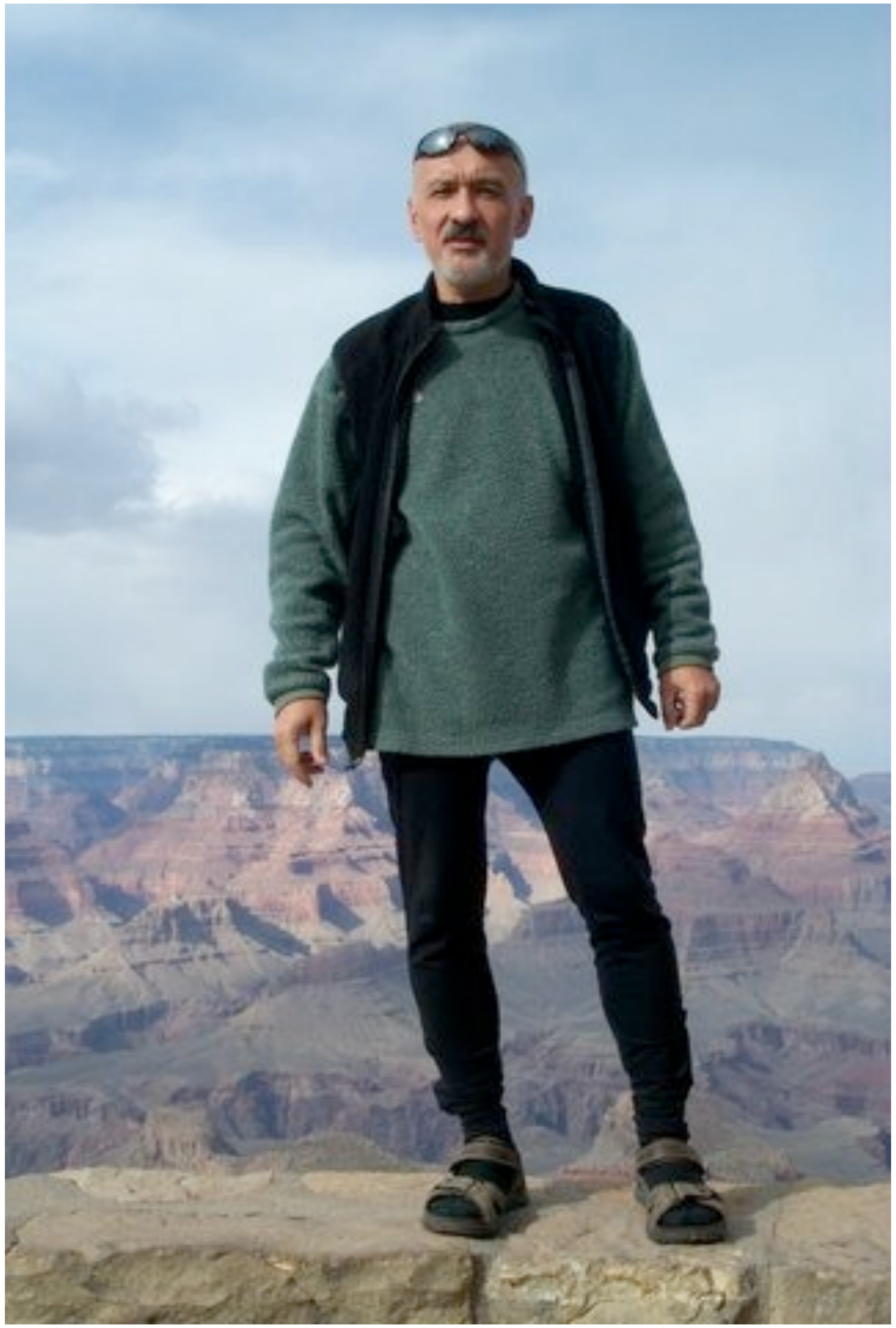
To absent friends...



Introduction

Plan of the talk

- Review of emergence of Einstein manifolds in the AdS/CFT correspondence
- Motivation for a possible relation between certain kinds of Einstein manifolds (including Sasaki-Einstein and 3-Sasaki) and triple systems

Based on:

arXiv:hep-th/9808014

with Bobby Acharya + Chris Hull + Bill Spence

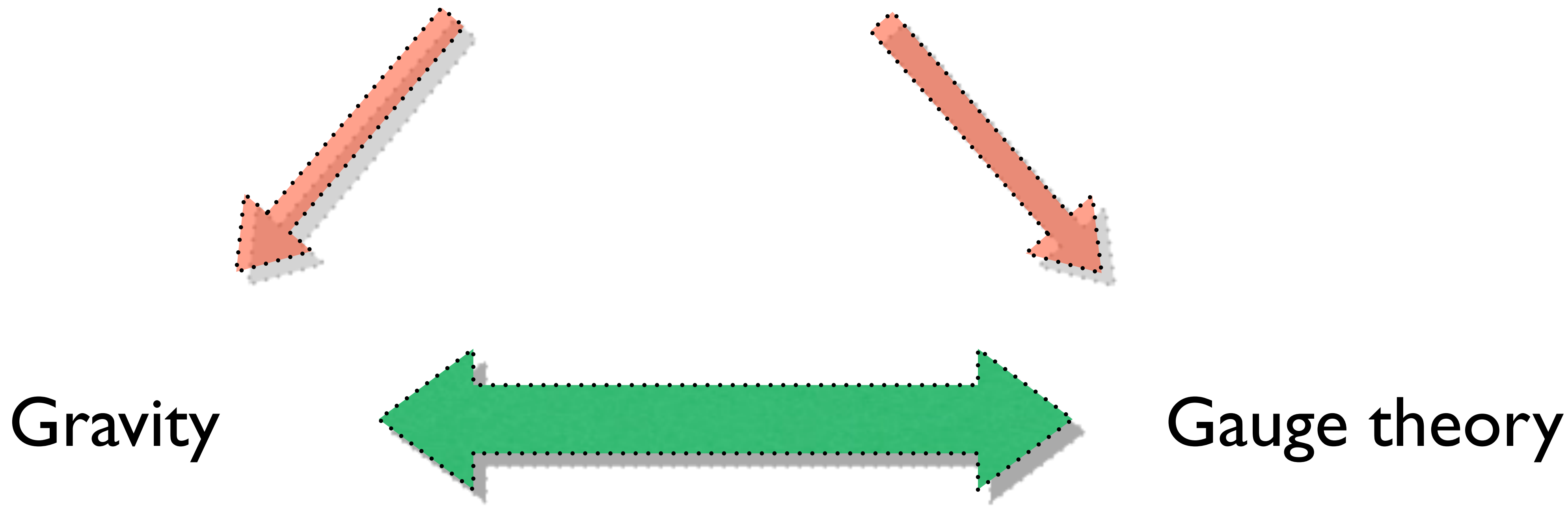
arXiv:0809.1086 [hep-th]

with Paul de Medeiros + Elena Méndez-Escobar
+ Patricia Ritter

and work/dream in progress

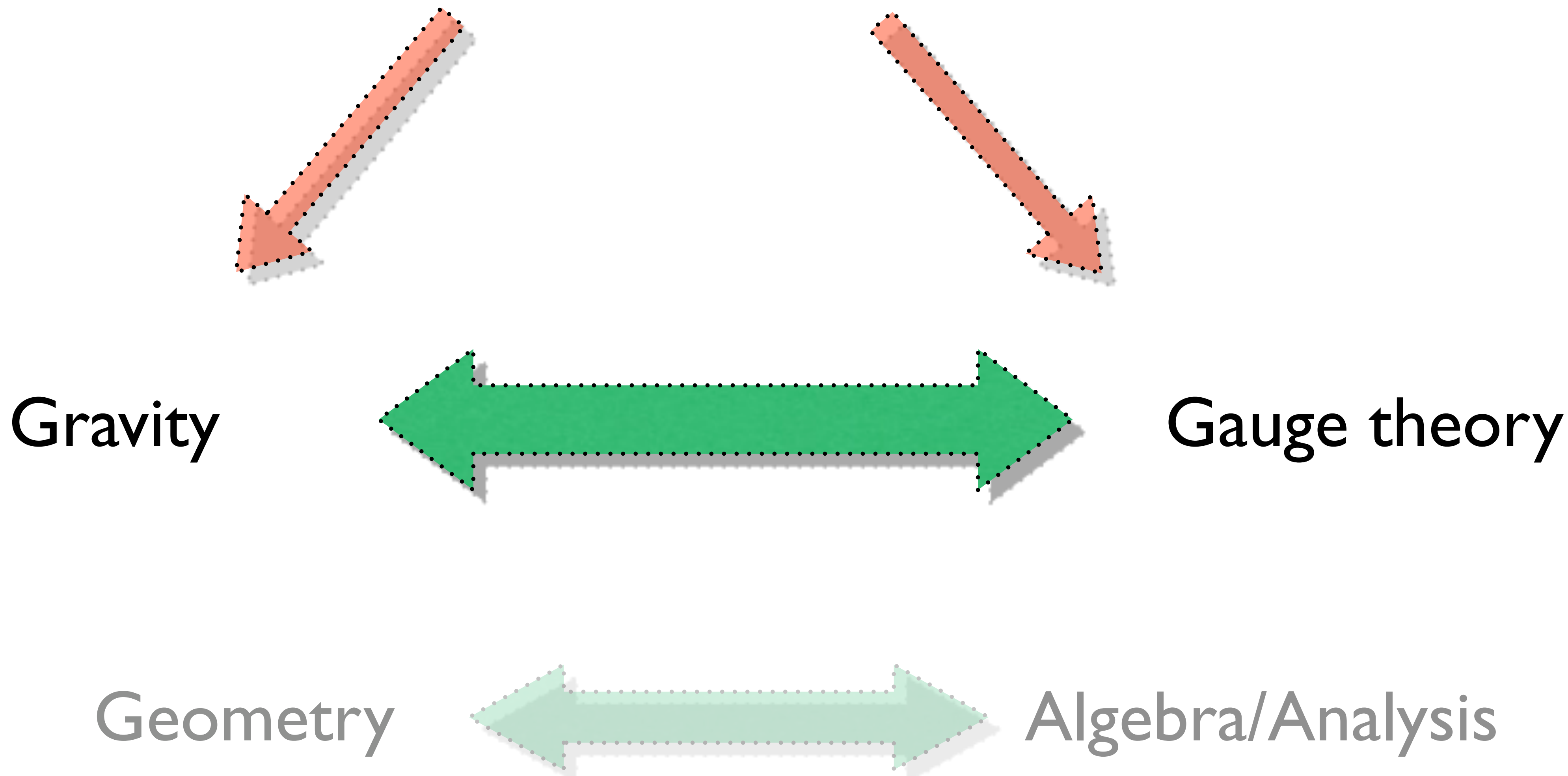
with Paul de Medeiros + Elena Méndez-Escobar

AdS/CFT

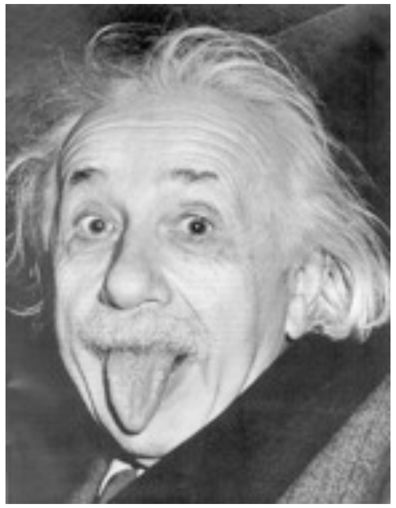


Maldacena (1997), Gubser+Klebanov+Polyakov (1998), Witten(1998)

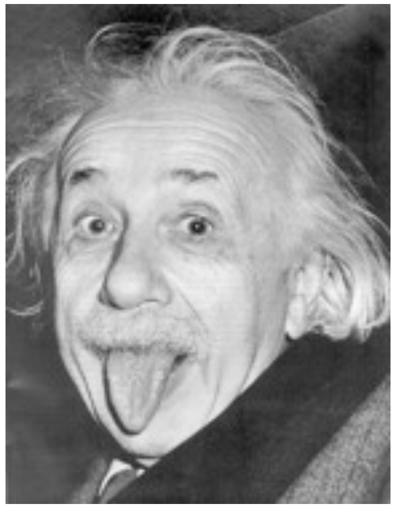
AdS/CFT



Maldacena (1997), Gubser+Klebanov+Polyakov (1998), Witten(1998)

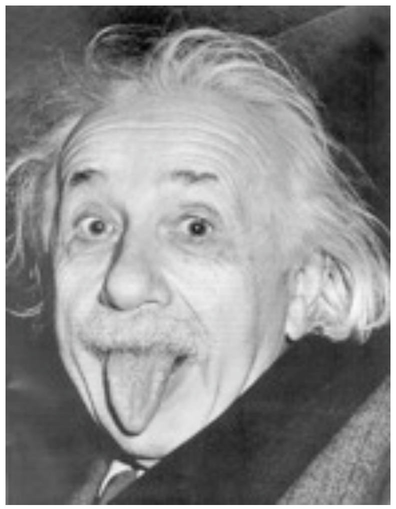


Einstein manifolds



Einstein manifolds

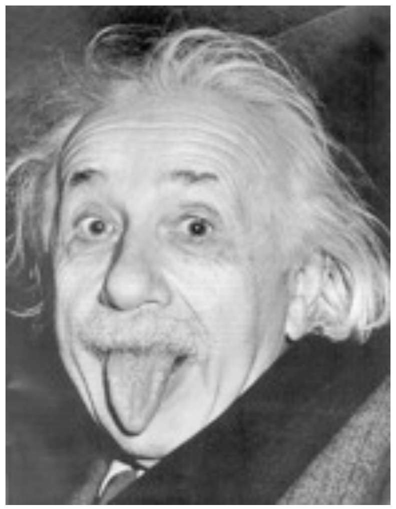
Triple systems



Einstein manifolds

Triple systems

7-dimensional
manifolds
admitting real Killing
spinors



Einstein manifolds

7-dimensional manifolds admitting real Killing spinors

Triple systems

Metric 3-Leibniz algebras

Einstein manifolds in AdS/CFT

M2-branes

Eleven dimensional supergravity admits a two-parameter family of half-supersymmetric backgrounds:

$$
\begin{aligned}
g &= H^{-2/3} g\left(\mathbb{R}^{1,2}\right) + H^{1/3}\left(dr^2 + r^2 g\left(S^7\right)\right) \\
F &= \mathrm{dvol}\left(\mathbb{R}^{1,2}\right) \wedge dH^{-1}
\end{aligned}
$$

where

$$
H = \alpha + \frac{\beta}{r^6}
$$

Duff+Stelle (1991)

For generic α and $\beta \propto n$, this describes a "stack" of n coincident **M2**-branes.

For $\beta = 0$, the background becomes (11-dimensional) Minkowski spacetime, whereas for $\alpha = 0$, it becomes

$$\mathrm{AdS}_4 \times S^7 \quad \text{with} \quad 2R_{\mathrm{AdS}} = R_S = \beta^{1/6}$$

which is the near-horizon geometry of the n coincident **M2**-branes.

Gibbons+Townsend (1993), Duff+Gibbons+Townsend (1994)

The isometry Lie algebra of a supergravity background extends to a Lie superalgebra: the **Killing superalgebra**.

For Minkowski spacetime it is the **Poincaré superalgebra**, but for the near-horizon limit of the M2-brane it is $\mathfrak{osp}(8|2)$.

The even Lie subalgebra is $\mathfrak{so}(8) \oplus \mathfrak{so}(2,3)$, which is the isometry Lie algebra of $\mathrm{AdS}_4 \times S^7$.

Generalisations

$$g = H^{-2/3} g(\mathbb{R}^{1,2}) + H^{1/3}(dr^2 + r^2 g(X^7))$$

$$F = \mathrm{dvol}(\mathbb{R}^{1,2}) \wedge dH^{-1}$$

Any Einstein 7-manifold, admitting real Killing spinors:

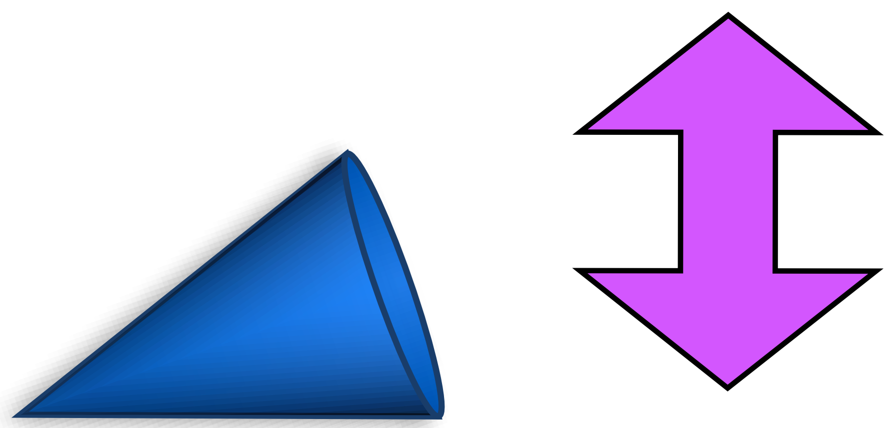
$$\nabla_V \phi = \tfrac{1}{2} V \cdot \phi$$

Interpretation: M2-branes at a conical singularity in a special holonomy 8-manifold

Real Killing spinors

The cone construction solves the problem of which riemannian manifolds admit real Killing spinors:

(X, g) admits real Killing spinors



$(\mathbb{R}^+ \times X, dr^2 + r^2 g)$ admits parallel spinors

Bär (1993)

If X is complete, then the cone is either flat or irreducible.

Gallot (1979)

Holonomy	Parallel spinors
Spin(7)	(1,0)
SU(4)	(2,0)
Sp(2)	(3,0)
{1}	(8,8)

Wang (1989)

7-manifolds with real Killing spinors

7-dimensional geometry	Holonomy of cone	Killing spinors
Weak G2 hol	Spin(7)	1
Sasaki-Einstein	SU(4)	2
3-Sasaki	Sp(2)	3
Sphere	$\{1\}$	8

Killing superalgebras

7-dimensional geometry	Killing superalgebra
Weak G2 hol	$\mathfrak{osp}(1,2)$
Sasaki-Einstein	$\mathfrak{osp}(2,2)$
3-Sasaki	$\mathfrak{osp}(3,2)$
Sphere	$\mathfrak{osp}(8,2)$

FO (1999)

Other cases may be obtained by quotienting the sphere.

There are **MANY** regular quotients of the round 7-sphere, all of which admit a spin structure and some even admit real Killing spinors.

FO+Gadhia (2006,?)

In this way one may also obtain $\mathfrak{osp}(N,2)$, for $N \leq 6$.

AdS/CFT predicts the existence of a three-dimensional superconformal field theory, whose symmetry superalgebra is isomorphic to the Killing superalgebra, but **reinterpreted**.

The even subalgebra is isomorphic to

$$\mathfrak{so}(2,3) \oplus \mathfrak{so}(N)$$

conformal algebra → $\mathfrak{so}(2,3)$

R-symmetry → $\mathfrak{so}(N)$

Chern-Simons theories

It took a decade to construct candidate theories realising these superconformal algebras.

I will not write them down here, but they are constructed by coupling Chern-Simons theory to matter hypermultiplets ***not*** (necessarily) in the adjoint representation.

They can be formulated succinctly in terms of certain **triple systems**, known as **metric 3-Leibniz algebras**.

Metric Leibniz algebras

A **(left) Leibniz** algebra is a Lie algebra without the skewsymmetry of the bracket:

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

Loday (1992)

It is said to be **metric** if it admits an invariant symmetric inner product:

$$\langle [X,Y],Z\rangle = -\langle Y,[X,Z]\rangle$$

Metric 3-Leibniz algebras

These are **ternary** versions of metric Leibniz algebras. They obey a **fundamental identity**:

$$[X, Y, [Z_1, Z_2, Z_3]] = [[X, Y, Z_1], Z_2, Z_3] + [Z_1, [X, Y, Z_2], Z_3] + [Z_1, Z_2, [X, Y, Z_3]]$$

and possess an invariant symmetric inner product:

$$\langle [X, Y, Z_1], Z_2 \rangle = - \langle Z_1, [X, Y, Z_2] \rangle$$

Geometric example

Any compact orientable 3-dimensional manifold defines one such algebra, albeit infinite-dimensional.

$$(M, \omega) \qquad \omega \in \Omega^3(M) \qquad f, g, h \in C^\infty(M)$$

$$df \wedge dg \wedge dh = [f, g, h]\omega \qquad [f, g, h] \in C^\infty(M)$$

$$\langle f, g \rangle = \int_M f g\, \omega$$

Nambu (1973)

Triple systems in AdS/CFT

Faulkner construction

$\mathfrak{k}, (-,-)$ a metric Lie algebra

V a representation

The transpose of the action gives a map:

$$R : V \times V^* \to \mathfrak{k}$$

defined by

$$(R(v,\alpha), X) = \alpha(X \cdot v)$$

This defines a trilinear map:

$$V \times V^* \times V \to V$$

by

$$[v, \alpha, w] = R(v, \alpha) \cdot w$$

Faulkner (1973)

In the special case when $V \cong V^*$ one gets a 3-bracket

$$V \times V \times V \to V$$

which obeys the fundamental identity of a 3-Leibniz algebra.

Orthogonal representations

$V, \langle -, - \rangle$ a real orthogonal representation

The 3-bracket $[x, y, z] = R(x, y) \cdot z$ given by

$$(R(x, y), X) = \langle X \cdot x, y \rangle$$

defines on V the structure of a metric 3-Leibniz algebra.

Not all metric 3-Leibniz algebras can be obtained in this fashion.

The tensor $\Omega(x,y,z,w) := \langle [x,y,z], w \rangle$

must obey the symmetry condition

$$\Omega(x,y,z,w) = \Omega(z,w,x,y)$$

since

$$\Omega(x,y,z,w) = (R(x,y), R(z,w))$$

However, this class includes **all** the ones appearing in superconformal Chern-Simons theories.

Unitarity of the quantum field theory requires that the inner product on V should have positive-definite signature.

This gives three types of constructions, depending on the type of the representation V :

real orthogonal, **complex** or **quaternionic** unitary representations.

Real reps

These include two well-known special cases:

- **3-Lie algebras**, where $[x,y,z]$ is totally skewsymmetric, and

Filippov (1985)

- **Lie triple systems**, where $[x,y,z]+[y,z,x]+[z,x,y]=0$

...,Jacobson (1951), Lister (1952), Yamaguti (1957)

Metric Lie triple systems are linear approximations to **riemannian symmetric spaces**.

$\mathfrak{g} = \mathfrak{k} \oplus V$ a 2-graded metric Lie algebra

The Lie bracket defines a 3-bracket

$$[x, y, z] = [[x, y], z]$$

relative to which $\boldsymbol{V}$ becomes a metric Lie triple system.

The map $R : \Lambda^2 V \to \mathfrak{k}$ given by the Lie bracket is the curvature operator of the symmetric space G/K.

To every metric 3-Lie algebra, Bagger+Lambert and Gustavsson constructed an $N=8$ superconformal Chern-Simons theory with superalgebra $\mathfrak{osp}(8|2)$.

Bagger+Lambert (2006,2007), Gustavsson (2007)

Unitarity requires the inner product to be positive-definite. **Alas** there is a unique (indecomposable, nontrivial) finite-dimensional positive-definite metric 3-Lie algebra.

FO+Papadopoulos (2002)

Nagy (2007)

Papadopoulos, Gauntlett+Gutowski (2008)

The general case of the real construction is a mixture of these two special cases and corresponds to 3-algebras introduced by Cherkis+Sämann in their study of superconformal Chern-Simons theories. (They had appeared earlier in work of Faulkner's.)

Faulkner (1973), Cherkis+Sämann (2008)

They generically give rise to $\mathcal{N}=1$ superconformal Chern-Simons theories.

Complex reps

For a complex unitary representation $\boldsymbol{V}$ one has

$$V^* \cong \overline{V}$$

and the corresponding 3-bracket is **sesquibilinear**

$$V \times \overline{V} \times V \to V$$

defining a class of hermitian 3-Leibniz algebras.

These include two well-known special cases:

- **N=6 triple systems**, where $[x,y,z]=-[z,y,x]$, and

Bagger+Lambert (2008)

- **hermitian Lie triple systems**, corresponding to **hermitian symmetric spaces**

The general case is a mixture of these two special cases and gives rise generically to $N=2$ superconformal Chern-Simons theories.

As the name indicates, the $N=6$ triple systems give rise to the $N=6$ Chern-Simons theories of Aharony, Bergman, Jafferis and Maldacena.

Aharony+Bergman+Jafferis+Maldacena (2008), Schnabl+Tachikawa (2008)

Embedding Lie (super)algebra

Both extreme cases of the complex Faulkner construction are characterised by the fact that they embed in a **3-graded metric complex Lie (super)algebra**:

$$\mathfrak{g}_{\mathbb{C}} = V \oplus \mathfrak{k}_{\mathbb{C}} \oplus \overline{V}$$

It is a Lie algebra in the case of the hermitian Lie triple systems and a Lie *super*algebra in the case of the $\boldsymbol{N=6}$ triple system.

Quaternionic reps

The 3-bracket is now defined using the complex symplectic structure ω on V:

$$R : S^2V \to \mathfrak{k}_{\mathbb{C}}$$

$$(R(x,y), X) = \omega(X \cdot x, y)$$

$$[x,y,z] = R(x,y) \cdot z$$

Quaternionic reps

The 3-bracket is now defined using the complex symplectic structure ω on V:

$$R : S^2V \to \mathfrak{k}_{\mathbb{C}}$$

$$(R(x,y), X) = \omega(X \cdot x, y)$$

$$[x,y,z] = R(x,y) \cdot z$$

This is the Hessian of the complex part of the **HKLR moment map** associated to the linear action of $\mathfrak{k}$ on V

These include two well-known special cases:

- **anti-3-Lie algebras**, where $[x,y,z]$ is totally symmetric, and

- **anti-Lie triple systems**, where $[x,y,z]+[y,z,x]+[z,x,y]=0$

The anti-3-Lie algebras correspond to **quaternionic Kähler symmetric spaces** and embed in a **5-graded complex metric Lie algebra**.

$$\mathfrak{g} = \mathbb{C}e \oplus V \oplus (\mathfrak{k}_{\mathbb{C}} \oplus \mathbb{C}h) \oplus V \oplus \mathbb{C}f$$

where e,f,h generate a $\mathfrak{sp}(1,\mathbb{C})$ Lie subalgebra and h is the grading element. The rest of the Lie bracket is given in terms of the complex symplectic form ω and the Faulkner map R.

The anti-Lie triple systems give rise to $\mathcal{N}=4$ and $\mathcal{N}=5$ superconformal Chern-Simons theories.

Gaiotto+Witten (2008), Hosomichi+Lee+Lee+Lee+Park (2008)

The difference between the $\mathcal{N}=4$ and $\mathcal{N}=5$ theories lies in the representation-theoretic content of the matter, once the R-symmetry is taken into account.

They also admit an embedding in a Lie superalgebra.

Dictionary

N	triple system	R-symmetry
1	real	
2	complex	$\mathfrak{u}(1)$
3	quaternionic	$\mathfrak{sp}(1)$
4	quaternionic anti-LTS	$\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$
5	quaternionic anti-LTS	$\mathfrak{sp}(2)$
6	complex BL4	$\mathfrak{su}(4)$
8	real 3-Lie	$\mathfrak{so}(8)$

Fantasy

Einstein 7-manifold X admitting real Killing spinors

Einstein 7-manifold X admitting real Killing spinors

A stack of n coincident **M2**-branes at apex of $C(X)$

Einstein 7-manifold $\boldsymbol{X}$ admitting real Killing spinors

A stack of $\boldsymbol{n}$ coincident **M2**-branes at apex of $\boldsymbol{C(X)}$

$\boldsymbol{d=11}$ SUGRA background $\mathrm{AdS}_4 \times X$ with $s \propto n^{-1/6}$

Einstein 7-manifold $\boldsymbol{X}$ admitting real Killing spinors

A stack of $\boldsymbol{n}$ coincident **M2**-branes at apex of $\boldsymbol{C(X)}$

$\boldsymbol{d=11}$ SUGRA background $\mathrm{AdS}_4 \times X$ with $s \propto n^{-1/6}$

Superconformal Chern-Simons+matter theory

Einstein 7-manifold X admitting real Killing spinors

A stack of n coincident **M2**-branes at apex of $C(X)$

$d{=}11$ SUGRA background $\mathrm{AdS}_4 \times X$ with $s \propto n^{-1/6}$

Superconformal Chern-Simons+matter theory

Metric 3-Leibniz algebra of dimension growing with n

This suggests that to every 7-dimensional Einstein manifold admitting real Killing spinors, there is associated a certain triple system; alternatively a metric Lie algebra and a faithful unitary representation.

The dimension of the triple system should grow as the manifold gets flatter. (cf. geometric quantisation)