Space of Ricci flows and its application
Based on a joint work with B. Wang

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Basic setup in Kähler Geometry

$(M, [\omega])$ is a polarized Kähler manifold where

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}\, d\,w_\alpha \wedge d\,\bar{w}_\beta > 0 \qquad \text{on } M.$$

In some local coordinate $U \subset M$, there is a local potential function ρ such that

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \rho}{\partial w_\alpha \partial \bar{w}_\beta}, \qquad \forall\ \alpha, \beta = 1, 2, \cdots n.$$

A Kähler class

$$[\omega] = \{\omega_\varphi \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } M\}$$

where φ is a real valued function.

Ricci form:

$$\begin{aligned} Ric(\omega) &= -\sqrt{-1}\partial\bar{\partial} \log \omega^n \\ &= -\sqrt{-1}\partial\bar{\partial} \; \log \; \det \left(g_{\alpha\bar{\beta}} \right). \end{aligned}$$

Scalar curvature:

$$\begin{aligned} R &= -g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial w_\alpha \bar{\partial} w_\beta} \log \det \left(g_{\alpha\bar{\beta}} \right) \\ &= -\triangle_g \log \det \left(g_{\alpha\bar{\beta}} \right). \end{aligned}$$

The first Chern class is positive definite (resp: negative definite) if

$$[Ric(\omega)] > (\text{resp.} <) \; 0 \;\; \text{on } M.$$

Calabi's program

$$Ca(\omega_\varphi) = \int_M \; (R(\omega_\varphi) - \underline{R})^2 \; \omega_\varphi^n.$$

- Mininmize this functional in each Kähler class.

- Critical points
  - Extremal Kähler metric: $g^{\alpha\bar{\beta}} \frac{\partial R}{\partial \bar{w}_\beta} \frac{\partial}{\partial w_\alpha}$ is holomorphic.
  - Constant Scalar curvature metric: $R(\omega_\varphi) = const.$
  - Kähler-Einstein metric: $Ric(\omega) = \lambda\omega$; where $\lambda = 1, 0, -1$.

Conjecture on KE metrics

E. Calabi: Does every Fano manifold with vanishing holomorphic vector fields admits KE metrics.

Yau: The existence of KE metrics in Fano manifold is related to the stability of the underlying polarization.

Yau-Tian-Donaldson conjecture: In algebraic manifold $(M, [\omega])$, the existence of cscK metric is equivalent to the K stability of $(M, [\omega])$.

An algebraic polarization is called **K stable** if the generalized futaki invariant in the central fiber in any non-trivial test configuration is negative.

This is first introduced by G. Tian in 1997 in terms of special degeneration and extended for more general setting by Donaldson.

The existence of KE metrics

- 1976

  - $C_1 = 0$, S. T. Yau, Calabi-Yau metric.

  - $C_1 < 0$, Existence of Kähler-Einstein metric,
    S. T. Yau and T. Aubin independently.

- 1988, $C_1 > 0$ and $n = 2$, G. Tian, Kähler-Einstein metric exists if and only if the Futaki invariant vanishes.

- 2004, X. Zhu + X. J. Wang, Fano toric manifold, Kähler-Einstein metric exists if and only if the Futaki invariant vanishes.

- 2007, X. Chen, C. LeBrun and B. Weber: There is a conformal Einstein metric in every Fano surface.

- 2008, S. K. Donaldson, In Toric Kähler surface, the existence of cscK metric if and only if it is K stable among toric invariant Kähler metrics.

Ricci flow

In 1982, R. Hamilton introduced the so called Ricci flow:

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(g(t)).$$

He subsequently proved that any 3 manifold with positive Ricci curvature can be deformed into a standard S^3 via Ricci flow.

On the Kähler Ricci flow

In canonical Kähler class with $C_1 > 0$, define

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \qquad \forall\, i,\ j = 1, 2, \cdots, n.$$

Evolution equation on curvatures:

$$\begin{aligned}\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} \;=\; & \triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{l}} \\ & + R_{i\bar{l}p\bar{q}} R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} - \frac{1}{2}\Big(R_{i\bar{p}} R_{p\bar{j}k\bar{l}} \\ & + R_{p\bar{j}} R_{i\bar{p}k\bar{l}} + R_{k\bar{p}} R_{i\bar{j}p\bar{l}} + R_{p\bar{l}} R_{i\bar{j}k\bar{p}}.\end{aligned}$$

and

$$\begin{cases} \frac{\partial}{\partial t} R_{i\bar{j}} = \triangle R_{i\bar{j}} + R_{l\bar{k}} R_{i\bar{j}k\bar{l}} - R_{i\bar{k}} R_{k\bar{j}}, \\ \frac{\partial}{\partial t} R = \triangle R + |\mathrm{Ric}|^2 - R. \end{cases}$$

Brief remark on the Kähler Ricci flow

- H. D. Cao proved the flow exists globally
- S. Bando (dimension $n = 3$), N. Mok (general dimension) proved that the positive bisectional curvature is preserved under the Kähler Ricci flow.
- (X.X. Chen + G. Tian, 2000) On any KE manifold, if the initial metric has positive bisectional curvature, then the Kähler Ricci flow will converge exponentially fast to a KE metric with constant bisectional curvature.
- Perelman announced a proof of the convergence of KR flow on KE manifold. A written proof appear in JAMS by X. Zhu + G. Tian

Conjecture 1 *(Hamilton-Tian) For Kähler Ricci flow initiated from any Kähler metric, it converge sequentially in Cheeger-Gromov sense to some KRS, with perhaps a different complex structure, except a codimension 4 singularity.*

Theorem 1 *Hamilton-Tian's conjecture holds in dimension 2.*

Theorem 2 *Suppose* $\{(M^n, g(t)), 0 \leq t < \infty\}$ *is a solution. If for every sequence* $t_i \to \infty$*, by passing to subsequence, we have (M, g(*t_i*))*$(\widehat{M}, \widehat{g})$*, where* $(\widehat{M}, \widehat{g})$ *is a KRS with codimension 4 singularities and the singularities is "nice". Then this flow is tamed.*

Theorem 3 *Suppose* $\{(M^n, g(t)), 0 \leq t < \infty\}$ *is a tamed Kähler Ricci flow solution. If* $\alpha_{\nu,1} > \frac{n}{(n+1)\nu}$*, then* φ *is uniformly bounded along this flow. In particular, this flow converges to a KE metric exponentially fast.*

A well known fact (for comparison)

Theorem 4 *If Tian's* $\alpha-$ *invariant* $> \frac{n}{n+1}$*, then the Kähler Ricci flow converges to KE metric exponentially fast.*

In view of these theorems, we need to check the following two conditions:

- Whether the Kähler Ricci flow is a tamed flow;
- Whether the local α-invariants are big enough.

The second condition can be checked by purely algebraic geometry method. The first condition is much weaker and we conjecture it always hold.

Theorem 5 *In Fano surface, the Kähler Ricci flow converges to a KRS.*

Perelman's Canonical Neighborhood theorem

Theorem 6 *Suppose* $\{(M^3, g(t)), 0 \leq t < T < \infty\}$ *is a Ricci flow solution which has a non-collasping constant* κ*. For every* ϵ*, there is an* $r_0(\epsilon, \kappa)$ *such that if* $Q = |Rm|(x_0, t_0) > r_0^{-2}$ *, then the ball* $B_{g(t_0)}(x_0, \epsilon^{-1} Q^{-\frac{1}{2}})$ *is diffeomorphic (by* ϕ*) to a geodesic ball* $B_{\tilde{g}(0)}(y, \epsilon^{-1} Q^{-\frac{1}{2}})$ *in a* κ*-solution* $\{(\tilde{M}, \tilde{g}), -\infty < t \leq 0\}$*. Moreover, up to the diffeomorphism* ϕ*, the difference between* $g(t + t_0)$ *and* $\tilde{g}(t)$ *in* C^k *topology is less than* ϵ *for every* $k \in [\frac{1}{\epsilon}], -\epsilon^{-2} Q < t \leq 0$*.*

A Ricci flow solution $\{(M, g(t)) | -\infty < t \leq 0\}$ is called a κ solution if it satisfies:

1. Ancient.

2. Curvature operator nonnonegative.

3. κ-noncollapsed.

4. M is a complete manifold.

5. $R > 0$ strictly. (This is the same thing as nonflat.)

What is ancient solution of the Kähler Ricci flow? is its moduli space compact?

Weak compactness

Theorem 7 *Let* $\mathcal{M}(\nu, \delta, \Lambda, C_S, n)$ *be the set of Kähler Einstein metrics* (M^n, g) *with*

- $\mathrm{Vol}_g(M) \geq \nu$,
- $\mathrm{Diam}_g(M) \leq \delta$,
- $\int_M |\,\mathrm{Rm}_g\,|^{\frac{n}{2}} \leq \Lambda$

Then $\mathcal{M}$ *is precompact in the Gromov-Hausdorff topology, and is compactified by including KE orbifolds that satisfy the same conditions.*

Form of the equations

Einstein case: [Anderson], [BKN], [Tian]

$$\triangle \, \mathrm{Rm} \;=\; \mathrm{Rm} * \mathrm{Rm}$$

For clarity we will consider mainly the Einstein equation; in rough form this looks like

$$\triangle u \;\geq\; -u^2$$

where $u = |\,\mathrm{Rm}\,|$.

Ordinarily one assumes the scale-invariant quantity $\int |\,\mathrm{Rm}\,|^{\frac{n}{2}}$ is bounded.

Consider $\triangle u \geq -fu$. If one may use the Sobolev inequality, then

$$f \in L^{\frac{n}{2}} \Rightarrow u \in L^q, \text{ all } q > 2$$
$$f \in L^p, \text{ some } p > \frac{n}{2} \Rightarrow u \in L^\infty$$

With $\triangle u \geq -u^2$, this theorem can't give us *pointwise* bounds on u.

Anderson (1989), BKN (1989), Tian (1990) managed to exploit the equation's *non*-linearity to partially recover the L^∞ bounds.

ϵ-Regularity

Theorem

There exists numbers ϵ_0, C so that

$$\int_{B(o,r)} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \;\leq\; \epsilon_0$$

implies

$$\sup_{B_{r/2}} |\,\mathrm{Rm}\,| \;\leq\; C\, r^{-2} \left(\int_{B_r} |\,\mathrm{Rm}\,|^{\frac{n}{2}} \right)^{\frac{2}{n}}.$$

The numbers ϵ and C depend on the Sobolev constant.

Proof: Modified version of Moser iteration.

What happen to ricci flow?

G. Perelman has the following fundmental theorem regarding KRF.

- the scalar curvature functional is uniformly bounded;

- the volume ratio is uniformly bounded from below;

- the Diameter is uniformly bounded from below;

With an observation by E. Calabi, we have: L^2 norm of Riem curvature is uniformly bounded from above along the Kähler Ricci flow.

Mimicing the Einstein metrics' case, we study the moduli sapce of space-time $\{(X^m, g(t)), -1 \leq t \leq 1\}$ satisfying the following conditions.

- $\frac{\partial g(t)}{\partial t} = -Ric_{g(t)} + c_0 g(t)$ where c_0 is a constant satisfying $0 \leq c_0 \leq c$.

- $\sup\limits_{X \times [-1,1]} |R|_{g(t)} \leq \sigma$.

- $\dfrac{\mathrm{Vol}_{g(t)}(B_{g(t)}(x,r))}{r^m} \geq \kappa$ for all $x \in X$, $t \in [-1,1]$, $r \in (0,1]$.

- $\int_X |Rm|_{g(t)}^{\frac{m}{2}} d\mu_{g(t)} \leq E$ for all $t \in [-1,1]$.

We denote such a moduli space as $M(m, c, \sigma, \kappa, E)$. Similar to the case of Einstein metrics, we are able to show the following weak compactness theorem.

Theorem 8 *If* $\{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\} \in M(m, c, \sigma, \kappa, E)$ *for every* i*, by passing to subsequence, we have* $(\mathsf{X}_i, x_i, g_i(0))(\hat{X}, \hat{x}, \hat{g})$ *for some* C^0*-orbifold* $\hat{X}$ *in Cheeger-Gromov sense. If* m *is odd, then* $\hat{X}$ *is a smooth manifold.*

In order to obtain Theorem 8, we need two essential estimates: local volume ratio upper bound and ϵ-regularity, i.e.,

$$\sup_{B_{g(0)}(p,\frac{\rho}{2})} |\nabla Rm|\rho^2 \leq C\{\int_{B_{g(0)}(p,\rho)} |Rm|_{g(0)}^{\frac{m}{2}}\}^{\frac{2}{m}}$$

whenever $\int_{B_{g(0)}(p,\rho)} |Rm|_{g(0)}^{\frac{m}{2}} < \epsilon$.

A sequence of spacetime $\{(X_i^m, g_i(t)), -1 \leq t \leq 1\}$ is called a refined sequence if the following properties are satisfied for every i.

1. $\frac{\partial g_i}{\partial t} = -Ric_{g_i} + c_i g_i$ and $\lim_{i \to \infty} c_i = 0$.

2. Scalar curvature norm tends to zero:

$$\lim_{i \to \infty} \sup_{(x,t) \in X_i \times [-1,1]} |R|_{g_i(t)}(x) = 0.$$

3. For every r, there exists $N(r)$ such that $(X_i, g_i(t))$ is κ-noncollapsed on scale r for every $t \in [-1, 1]$ whenever $i > N(r)$.

4. Energy uniformly bounded by E:

$$\int_{X_i} |Rm|_{g_i(t)}^{\frac{m}{2}} d\mu_{g_i(t)} \leq E, \qquad \forall\, t \in [-1, 1].$$

A refined sequence $\{(X_i, g_i(t)), -1 \leq t \leq 1\}$ is called an E-refined sequence if there exists a constant H such that

$$\int_{B(x, |Rm|_{g_i(t)}^{-\frac{1}{2}}(x))} |Rm|_{g_i(t)}^{\frac{m}{2}} d\mu_{g_i(t)} > \epsilon$$

whenever $(x, t) \in X_i \times [-\frac{1}{2}, 0]$ and $|Rm|_{g_i(t)}(x) > H$.

In short, an E-refined sequence is a refined sequence whose center-part-solutions satisfy energy concentration property.

An E-refined sequence $\{(X_i, g_i(t)), -1 \leq t \leq 1\}$ is called an EV-refined sequence if there is a constant K such that

$$\frac{\mathrm{Vol}_{g_i(t)} B_{g_i(t)}(x,r)}{r^m} < K$$

for every i and $(x,t) \in X_i \times [-\frac{1}{4}, 0]$, $r \in (0, 1]$.

In short, an EV-refined sequence is an E-refined sequence whose center-part-solutions have bounded volume ratios (from both sides).

Lemma 1 *Every refined sequence is an E-refined sequence.*

Lemma 2 *Every E- refined sequence is an EV-refined sequence.*

Lemma 3 Weak Compactness of an EV-refined Sequence in $C^{1,\gamma}$-topology *Suppose* $\{(X_i, x_i, g_i(t)), -1 \leq t \leq 1\}$ *is an EV-refined sequence,* $t_i \in [-\frac{1}{4}, 0]$. *Then* $(X_i, x_i, g_i(t_i))$ *converges to a Ricci-flat multifold* (X, x, g) *in Gromov-Hausdorff topology. Furthermore, there are* $L(\leq N_0)$ *points* $p^1, \cdots, p^L \in X$ *such that* $X \backslash \{\bigcup_{s=1}^{L} p^s\}$ *is smooth and the convergence is in* $C^{1,\gamma}$*-topology away from* $\{p^s\}_{s=1}^{L}$ *for any* $\gamma \in (0, 1)$. *For brevity, we denote this convergence as* $(X_i, x_i, g_i(t_i)) \stackrel{C^{1,\gamma}}{\to} (X, x, g)$.

The limit multifold (X, g) *satisfies the following estimates*

- *For every point* $p^s (1 \leq s \leq L)$, *the number of cone-like ends at* p^s *is bounded by* $\frac{2^m K}{\kappa}$, *i.e.,* $rank(H^0(X, X \backslash \{p^s\})) \leq \frac{2^m K}{\kappa}$.

- $\int_X |Rm|_g^{\frac{m}{2}} d\mu_g \leq E$.

In particular, (X, g) is a κ-noncollapsed, Ricci-flat ALE multifold.