Sasakian Geometry: Recent Work of Kris Galicki

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 $\Longleftrightarrow$  $\Longleftrightarrow$ 

Cone **Sasakian** **transverse**

Motivation: Einstein Metrics

1. Are Einstein manifolds scarce or numerous?

2. On a given manifold are there many or few Einstein metrics?

Sketch Proof: of existence of many positive Einstein metrics on many $2n+1$-dim'l manifolds ($n > 1$).

Main ingredients:

1. Contact geometry
2. Algebraic geometry

Main Reference: **C.P.B. and K. Galicki, Sasakian Geometry**, Oxford UP, 2008.

The metric g is **Einstein** if

$\mathrm{Ric}_g = \lambda g, \qquad \lambda$ constant

Three cases $\lambda > 0, \lambda < 0, \lambda = 0$. Here positive only $\lambda > 0$.

• **Contact Manifold**(compact)

A **contact 1-form** η *such that*

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a **contact structure**

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, *take* $f > 0$. *or equivalently a codimension 1 sub-bundle* $\mathcal{D} = \mathrm{Ker}\ \eta$ *of* TM.

$(\mathcal{D}, d\eta)$ *symplectic vector bundle*

Unique vector field ξ, *called the* **Reeb vector field**, *satisfying*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

The **characteristic foliation** $\mathcal{F}_\xi$ *each leaf of* $\mathcal{F}_\xi$ *passes through any nbd* U *at most* k *times* $\iff$ **quasi-regular**, $k = 1 \leftrightarrow$ *regular, otherwise* **irregular**

Quasi-regularity is strong, most contact 1-forms are irregular.

Contact bundle $\mathcal{D} \longrightarrow$ *choose* **almost complex structure** J *extend to* Φ *with* $\Phi\xi = 0$

Get a compatible metric

$$g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$$

Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ *called* **contact metric structure**

The pair $(\mathcal{D}, J)$ *is a strictly pseudo-convex almost CR structure.*

Question: *Which contact metrics are Einstein?*

Definition: *The structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *is* **K-contact** *if* $\pounds_\xi g = 0$ *(or* $\pounds_\xi \Phi = 0$*). It is* **Sasakian** *if in addition* $(\mathcal{D}, J)$ *is integrable.*

Note: Quasi-regular $\Rightarrow$ *K-contact.*

Thm: *(B-,Galicki) K-contact + Einstein* $\Rightarrow$ *Sasaki-Einstein with* $\lambda > 0$.

Remark: The only known contact metric that is Einstein and not Sasakian is the flat metric on 3-D torus T^3 *(or quotient thereof).*

Blair: this is the only flat contact metric.

orbifold Boothby-Wang: *Manifold* M *compact with* (ξ, η, Φ, g) *quasi-regular* $\Rightarrow$ *quotient* $\mathcal{Z} = M/\mathcal{F}_\xi$ *almost Kähler orbifold*

Converse: $\mathcal{Z} = M/\mathcal{F}_\xi$ almost Kähler orbifold. ω Kähler form with $[\omega] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. Total space M of S^1 orbibundle over $\mathcal{Z}$ has K-contact structure. $(\mathcal{Z}, \omega)$ is projective algebraic orbifold $\Longleftrightarrow$ (ξ, η, Φ, g) is Sasakian.

$(\mathcal{Z}, \omega)$ Kähler-Einstein (KE) with $\lambda > 0$ $\Longleftrightarrow$ (ξ, η, Φ, g) is Sasaki-Einstein (SE), $\mathrm{Ric}_g = 2ng$.

Question: *When do we have SE or KE metrics?*

1. $c_1^{orb}(\mathcal{Z}) > 0$ *(easy)*

2. solve Monge-Ampère equation (hard, continuity method)

$$\frac{\det(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j})}{\det(g_{i\bar{j}})} = e^{f - t\phi}.$$

Tian: uniform boundedness

$$\int_{\mathcal{Z}} e^{-\gamma t \phi_{t_0}} \omega_0^n < +\infty$$

Many people Yau, Tian, Siu, Nadel, and most recently by Demailly and Kollár in orbifold category.

alebraic geometry *of* **orbifolds***:*

local unif covers, ram index: m_j

branch divisor: $\mathbb{Q}$-divisor

$$\Delta := \sum (1 - \frac{1}{m_j}) D_j$$

canonical orbibundle

$$K_{\mathcal{Z}}^{orb} = K_{\mathcal{Z}} + \sum (1 - \frac{1}{m_j})[D_j],$$

Kawamata log terminal (klt): For every $s \geq 1$ and holomorphic section $\tau_s \in H^0(\mathcal{Z}, \mathcal{O}((K_{\mathcal{Z}}^{orb})^{-s})$ there is $\gamma > \frac{n}{n+1}$ such that $|\tau_s|^{-\gamma/s} \in L^2(\mathcal{Z})$.

Theorem 2: $c_1^{orb}(\mathcal{Z}) > 0$, klt $\Rightarrow$ Sasaki-Einstein metric.

How to find them:

1. Links of weighted homogeneous polynomials. C.B., Galicki, Kollár, Nakamaye

Sasakian Geometry of Links

$\mathbb{C}^{n+1}$ coordinates $\mathbf{z} = (z_0, \ldots, z_n)$
weighted $\mathbb{C}^*$-action

$$(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n),$$

weight vector $\mathbf{w} = (w_1, \cdots, w_n)$
with $w_j \in \mathbb{Z}^+$ and
$\gcd(w_0, \ldots, w_n) = 1$.
f weighted homogeneous polynomial

$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n)$$

$d \in \mathbb{Z}^+$ is **degree** of f.

$0 \in \mathbb{C}^{n+1}$ isolated singularity.

link L_f defined by

$$L_f = f^{-1}(0) \cap S^{2n+1},$$

S^{2n+1} unit sphere in $\mathbb{C}^{n+1}$

Special Case: Brieskorn-Pham poly. (BP)

$$f(z_0, \ldots, z_n) = z_0^{a_0} + \cdots + z_n^{a_n}$$

Fact: L_f has natural Sasakian structure with commutative diagram: Sasaki, Abe, Takahashi, C.B, Galicki

$$\begin{array}{ccc} L_f & \longrightarrow & S^{2n+1}_{\mathbf{w}} \\ \Big\downarrow \pi & & \downarrow \\ \mathcal{Z}_f & \longrightarrow & \mathbb{P}_{\mathbb{C}}(\mathbf{w}), \end{array}$$

horizontal arrows: Sasakian and Kählerian embeddings.

vertical arrows: orbifold Riemannian submersions.

$S^{2n+1}_{\mathbf{w}}$ weighted sphere

$\mathbb{P}_{\mathbb{C}}(\mathbf{w})$ weighted projective space

Topology of Links

Milnor Fibration Theorem $\Rightarrow L_f$ is (n-2)-connected. So topology is determined by $H_{n-1}(L_f, \mathbb{Z})$.

monodromy map h_* induced by $S^1_{\mathbf{w}}$ action $\Rightarrow$ Alexander polynomial $\Delta(t) = \det(t\mathbb{I} - h_*) \to b_{n-1}(L_f)$ = # of factors of $(1-t)$ in $\Delta(t)$

Torsion: Orlik conjecture: Holds for BP's, and generally for dimensions 3 and 5.

BP: spheres– Brieskorn Graph Thm.

Sasaki-Einstein metrics

Positivity $\Rightarrow$ $I = (\sum w_i - d) > 0$

klt estimates for L_f

$$d(\sum w_i - d) < \frac{n}{n-1} \min_{i,j} w_i w_j.$$

BP polyn: (better)

$$1 < \sum_{i=0}^{n} \frac{1}{a_i} < 1 + \frac{n}{n-1} \min_i \{\frac{1}{a_i}, \frac{1}{b_i b_j}\}.$$

a_i BP exponents and

$$b_i = \gcd(a_i, \mathrm{lcm}(a_j \mid j \neq i))$$

$\exists$ other estimates. Positivity plus a klt estimate $\Rightarrow$ SE metric

SE Moduli: infinitesimal deformations

Thm L_f a link with index $I > 0$ satisfying any klt estimate. Then L_f admits a μ_{SE}-dimensional family of SE metrics where μ_{SE} equals

$$2[h^0(\mathbb{CP}(\mathbf{w}), \mathcal{O}(d)) - \sum_i h^0(\mathbb{CP}(\mathbf{w}), \mathcal{O}(w_i))$$

$$+ \dim\ \mathfrak{Aut}(\mathcal{Z}_f)]$$

$N_{SE} = \#$ of deformation classes SE metrics.

Obstructions

Only known topological obstructions to Einstein metrics occur in dimension 4. Hitchin-Thorpe and more LeBrun

There are obstructions to SE metrics: Gauntlett,Martelli,Sparks,Yau

Estimate of Lichnerowicz $\Rightarrow$ if $I > n$ $\mathrm{min}_i w_i$ then $\nexists$ SE metrics.

Only applies to KE orbifolds!

Cases when estimate is sharp: (Ghigi-Kollár) $\exists$ SE metric $\Longleftrightarrow$ $I < n$ $\mathrm{min}_i w_i$ only homotopy spheres.

The Results

Homotopy Spheres:Kervaire,Milnor

$bP_{2n} =$ group of homotopy spheres S^{2n-1} that bound a parallelizable manifold. $bP_8 = \mathbb{Z}_{28}$, $bP_{12} = \mathbb{Z}_{992}$, $bP_{16} = \mathbb{Z}_{8128}$, $bP_{4n+2} = 0$ or $\mathbb{Z}_2$; $bP_6 = bP_{14} = bP_{30} = 0$.

(B,Galicki,Kollár)

• Each 28 diffeo types of S^7 admits hundreds of SE metrics. Largest $\mu_{SE} = 82$, standard S^7.

• All 992 diffeo types in bP_{12} and all 8128 diffeo types in bP_{16} admit SE metrics.

• All elements of bP_{4n+2} admit SE metrics.

Conjecture: All elements of bP_{2n} admit SE metrics.

• Both N_{SE} and μ_{SE} grow double exponentially with dimension.

Reason for growth: Sylvester's sequence determined by $c_{k+1} = 1+c_0 \cdots c_k$ begins as $2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$

Sequences $\mathbf{a} = (a_0 = c_0, \ldots, a_{n-1} = c_{n-1}, a_n)$ with $c_{n-1} < a_n < c_0 \cdots c_{n-1}$ give SE metrics

Examples:

(1) $N_{SE}(S^{13}) > 10^9$ and $\mu_{SE}(S^{13}) = 21300113901610$

(2) $N_{SE}(S^{29}) > 5 \times 10^{1666}$ and $\mu_{SE}(S^{29}) > 2 \times 10^{1667}$

Conjecture: Both $N_{SE}(S^{2n-1})$ and $\mu_{SE}(S^{2n-1})$ are finite.

Sasaki Geometry Dimension 5

Smale-Barden classification: 5-manifolds M^5 with $\pi_1(M^5)$ trivial. Assume spin $\Rightarrow$ Smale

BUILDING BLOCKS

$$M_\infty = S^2 \times S^3, \; M_1 = S^5$$

$$M_j, \; H_2(M_j, \mathbb{Z}) = \mathbb{Z}_j \oplus \mathbb{Z}_j$$

$$M^5 = M_{k_1} \# \cdots \# M_{k_s},$$

k_i divides k_{i+1} or $k_{i+1} = \infty$.

Which M^5 admit Sasakian structures?

Not known generally. However, Kollár: write (p prime)

$$H_2(M^5, \mathbb{Z}) = \mathbb{Z}^k \oplus \bigoplus_{p,i} (\mathbb{Z}_{p^i})^{c(p^i)}$$

Sasakian $\Rightarrow$

$\#\{i \mid c(p^i) > 0\} \leq k + 1$

Example: $M_p^5 \# M_{p^2}^5$ is **not** Sasakian, but both M_p^5 and $M_{p^2}^5$ are.

M^5 Sasakian $\Rightarrow$
$H_2(M^5, \mathbb{Z}) = \mathbb{Z}^k \oplus \Sigma_i(\mathbb{Z}_{m_i})^{2g(D_i)}$
where D_i branch divisor = Riemann surface genus $g(D_i)$ Kollár

Assume positive Sasakian: Kollár:
$H_2(M^5, \mathbb{Z})_{\text{tor}}$ must be
$\mathbb{Z}_m^2, \ \mathbb{Z}_2^{2n}, \ \mathbb{Z}_3^4, \ \mathbb{Z}_3^6, \ \mathbb{Z}_3^8, \ \mathbb{Z}_4^4, \ \mathbb{Z}_5^4$
Note $m = 1 \Rightarrow$ no torsion. All such groups occur, but not all rat'l homology spheres. $M_\infty \# M_{30k}$ admits SE metric, but M_{30k} does not! All the others do.

(B-,Galicki,Nakamaye; Kollár): nM_∞ all admit SE metrics $N_{SE}(nM_\infty) = \infty \ \forall \ n > 0$.

Rational homology spheres (π_1 trivial) (B-,Galicki; Kollár) with $c_1^{orb} > 0 \iff$ $nM_2, n > 0, \ M_m, \ m \neq 30k,$ $2M_3, 3M_3, 4M_3, 2M_4, 2M_5$ all known to admit SE except nM_2 with $n \geq 8$ or $n = 4$.

$N_{SE}(2M_5) = N_{SE}(4M_3) = 1$

$\mu_{SE}(2M_5) = 6, \ \mu_{SE}(4M_3) = 14$

$N_{SE}(2M_4) = 2$ (Kollár)

Mixed types: $kM_\infty \# N$, $k > 0, m > 1$; $N =$ a rat'l homology sphere as above, but now we can have $m = 30l$.

$m \geq 12 \Rightarrow k \leq 8$ BGK

$kM_\infty \# M_m$ where $1 \leq k \leq 8$ and $m \geq 12$ have SE metrics, and

$N_{SE}(M_\infty \# M_m) = 4$;

$N_{SE}(2M_\infty \# M_m) = 3$;

$N_{SE}(kM_\infty \# M_m) = 2,\ k = 3, 4$;

$N_{SE}(kM_\infty \# M_m) = 1, k = 5, \ldots, 8.$

$N = 4M_3,\ 2M_5 \Rightarrow k = 0.$

$N = 2M_4 \Rightarrow k = 0, 1.$ (Kollár)

- $M_\infty \# M_m$ has SE metrics for $m = 2, \cdots, 7$.
- $M_\infty \# n M_3$ has SE metrics for $n = 2, 3$.
- $2M_\infty \# M_m$ has SE metrics for $m = 2, \cdots, 5$.
- $3M_\infty \# M_m$ has SE metrics for $m = 2, 3, 4, 7, 9, 10$.
- $4M_\infty \# M_m$ has SE metrics for $4 \neq m \geq 2$.
- $4M_\infty \# 2M_2$ has SE metrics.
- $5M_\infty \# M_m$ has SE metrics for $m = 2$. (B-,Nakamaye)

- $5M_\infty \# 2M_2$ has SE metrics
- $6M_\infty \# M_m$ has SE metrics for $m \geq 2$.
- $7M_\infty \# M_m$ has SE metrics for $m \geq 3$.
- $8M_\infty \# M_m$ has SE metrics for $m \geq 5$. (B-,Galicki)

SE unknown for $kM_\infty \# M_m$ with $k > 8$ and $2 \leq m < 12$.

Problem: Completeness of SE moduli. Relate to Einstein moduli (Koiso).

Higher Dimension

- Sasakian with pos Ricci curv on $2k(S^{2n-1} \times S^{2n})$ for different diffeo types. E.g. on all 28 diffeo classes for $2(S^3 \times S^4)$.
SE for $222(S^3 \times S^4)$ and $480(S^3 \times S^4)$ (C.B.,Galicki)

- SE metrics on $S^2 \times S^5$ (C.B.,Galicki,Ornea)

- Many others with torsion