- Integrable systems on metric graphs -

The KdV equation (preliminary results)

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Motivation (Leitmotiv)

Question:

Is there a theory of solitary waves on networks?

In particular:

Can one split a solitary wave into several solitary waves at junctions?

Possible applications in

Optical fibres

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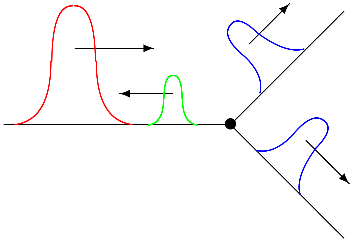
Motivation

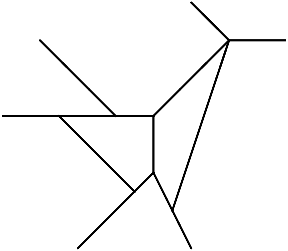
Example (A desideratum):



An incoming solitary wave from the left approaching a node
and then being split into 3 outgoing solitary waves
one of which is a reflected wave
and two of which are transmitted waves

Metric graphs

Definition: A metric graph $\mathcal{G}$ is a finite collection of half lines and intervals of given lengths with an identification of some of its endpoints (=vertices)



A graph with $n = 6$ external lines and $m = 8$ internal lines

$\mathcal{G}$ is a metric space:

There is the unique notion of a distance between two points

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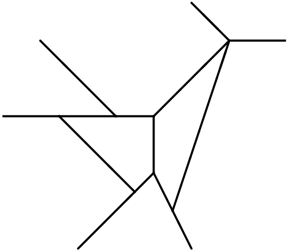

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$\mathcal{G}$ is a metric space:

There is the unique notion of a distance between two points

Analysis

It makes sense to speak of

1. Functions on the graph
2. Measureable functions
3. Lebesgue integration
4. Continuous functions
5. Contiouous functions away from the vertices
6. Differentiation
7. Differentiable functions on the graph
8. Differentiable functions away from the vertices
9. Infinitely differentiable functions away from the vertices and with compact support

Call the last set of functions (a linear space) $\mathcal{C}_c$.

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Dirac Operator

Forms on $\mathcal{G}$:

$$\mathsf{f} = f_0 + f_1 dx = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \quad f_0, f_1 \in \mathcal{C}_c$$

Differentiation d and Codifferentiation δ

$$f' = \frac{d}{dx} f, \quad f \in \mathcal{C}_c$$

$$d\mathsf{f} = 0 + f_0' dx = \begin{pmatrix} 0 \\ f_0' \end{pmatrix}$$

$$\delta\mathsf{f} = -f_1' + 0 dx = \begin{pmatrix} -f_1' \\ 0 \end{pmatrix}$$

Properties: $d^2 = 0, \qquad \delta^2 = 0$

Call the linear space of such f's $\Omega_c = \mathcal{C}_c \oplus \mathcal{C}_c$

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The Dirac operator is

$$\boxed{D = d + \delta}$$

which reads as

$$D\mathrm{f} = \begin{pmatrix} -f_1' \\ f_0' \end{pmatrix}$$

and hence the Dirac-Laplace operator is

$$D^2\mathrm{f} = (d\delta + \delta d)\mathrm{f} = -\begin{pmatrix} f_0'' \\ f_1'' \end{pmatrix}$$

Dirac Operator

On metric graphs the Dirac operator will play the role of

$$\frac{d}{dx}$$

Recall:

On the real line this is the infinitesimal generator of translations associated to a

conserved quantity

namely the

momentum

in

integrable systems

.

Boundary conditions

For any f $\in \Omega_c$ define

$$[\mathrm{f}] = \begin{pmatrix} [f_0] \\ [f_1] \end{pmatrix} \in \mathbb{C}^{2(n+2m)}$$

to be the set of boundary values at the vertices of the graph $\mathcal{G}$. n is the number of external lines (= half lines), m is the number of internal lines (=finite intervals). Consider boundary conditions of the form

$$A[f_0] + B[f_1] = 0$$

where A and B are $(n+2m) \times (n+2m)$ matrices. Call the resulting set of f's satisfying these linear conditions $\Omega_c(A, B)$. Based on results by Kostrykin, Schrader on Laplace operators on $\mathcal{G}$ there is

Theorem[Bolte, Harrison]: If the $(n+2m) \times 2(n+2m)$ matrix (A, B)

1. has maximal rank $= n + 2m$
2. and $AB^\dagger$ is hermitean,

then D restricted to $\Omega_c(A, B)$ is essentially selfadjoint on the Hilbert space of square integrable forms.

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Boundary conditions: An interpretation

Consider the map

$$\begin{array}{rccl} \tau: & \Omega_c & \to & \mathcal{C}_c \oplus \mathcal{C}_c \\ & \mathrm{f} & \mapsto & \tau(\mathrm{f}) = (f_0 + \mathrm{i} f_1, f_0 - \mathrm{i} f_1) \end{array}$$

such that

$$\tau(D\mathrm{f}) = \tau(D)\tau(\mathrm{f})$$

holds with

$$\tau(D) = \mathrm{i}\frac{d}{dx} \oplus -\mathrm{i}\frac{d}{dx}$$

According to von Neumann's theory of selfadjoint extensions involving the concept of deficiency indices

1. $\mathrm{i}\frac{d}{dx}$ has deficiency indices $(m, n+m)$
2. $-\mathrm{i}\frac{d}{dx}$ has deficiency indices $(n+m, m)$ and hence
3. $\mathrm{i}\frac{d}{dx} \oplus -\mathrm{i}\frac{d}{dx}$ has deficiency indices $(n+2m, n+2m)$

Differential algebra

Consider 2×2 matrices of the form

$$\mathsf{a} = \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix}, \qquad a_0, a_1 \in \mathcal{C}_c$$

This set is isomorphic to the set Ω_c of smooth forms and constitutes a commutative ring $\mathcal{R}_c$ under matrix multiplication (with complex multiples of a unit added).

Also Ω_c is a module of this ring via

$$\mathsf{af} = \begin{pmatrix} a_0 & -a_1 \\ a_1 & a_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} a_0 f_0 - a_1 f_1 \\ a_1 f_0 + a_0 f_1 \end{pmatrix}$$

The Dirac operator defines a map on this ring also denoted by D

$$D\mathsf{a} = \begin{pmatrix} -a_1' & -a_0' \\ a_0' & -a_1' \end{pmatrix}$$

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Differential algebra

The following Leibniz rules are satisfied

$$D(\mathrm{af}) = (D\mathrm{a})\mathrm{f} + \mathrm{a}(D\mathrm{f}), \qquad D(\mathrm{ab}) = (D\mathrm{a})\mathrm{b} + \mathrm{a}(D\mathrm{b})$$

Following the approach to integrable systems initiated by Gel'fand, Dikii we therefore introduce the following ring of differential operators

$$\mathcal{R}_c[D] = \left\{ \mathbf{X} = \sum_{n=0}^{d} \mathrm{x}_n D^n \;\middle|\; \mathrm{x}_n \in \mathcal{R}_c \right\}.$$

and the ring of pseudo-differential operators

$$\mathcal{R}_c\{D\} = \left\{ \mathbf{X} = \sum_{n=-\infty}^{d} \mathrm{x}_n D^n \;\middle|\; \mathrm{x}_n \in \mathcal{R}_c \right\},$$

objects first introduced by Schur (1904). d is called the order of $\mathbf{X}$.

Differential algebra

The composition rules are such that the following relations involving D^{-1} are valid

$$\begin{aligned} D^k \circ D^l &= D^{k+l}, \quad k, l \in \mathbb{Z} \\ D^{-1} \circ \mathsf{a} &= \sum_{k=0}^{\infty} (-1)^k (D^k \mathsf{a}) D^{-k-1}. \end{aligned} \qquad (1)$$

Properties

1. The kernel $\mathrm{Ker} D = \{\mathsf{a} \mid D\mathsf{a} = 0\}$ of D consists of those a whose entries are piecewise constant functions on $\mathcal{G}$
2. The set
$$\mathrm{Ker}\, D\{D\} = \left\{ \sum_{k=-\infty}^{n} \mathsf{a}_k D^k \mid \mathsf{a}_k \in \mathrm{Ker}\, D \right\}$$
is a commutative subring of $\mathcal{R}_c\{D\}$.
3. The center $\mathcal{Z}(\mathcal{R}_c\{D\})$ of $\mathcal{R}_c\{D\}$ equals $\mathrm{Ker}\, D$

Differential algebra

Definition: A Lax operator $\mathbf{L}$ is a differential operator, for which the coefficient of the leading order equals 1.

Example: $\mathbf{L} = D^2 + \mathrm{q}$ (KdV)

The following result in its classical form goes back to Schur

Theorem: Consider any pseudo differential operator $\mathbf{X}$, whose order equals d and whose coefficient of the leading order equals one. Then $\mathbf{X}$ has a unique dth root of the form

$$\mathbf{X}^{\frac{1}{d}} = D + \mathrm{b}_0 + \mathrm{b}_1 D^{-1} + \cdots$$

The commutator $\mathcal{Z}_{\mathcal{R}_c\{D\}}(\mathbf{X})$ of $\mathbf{X}$ in $\mathcal{R}_c\{D\}$ is the commutative ring of formal Laurent series

$$\sum_{k=-\infty}^{m} \mathrm{u}_k \mathbf{X}^{k/d}, \qquad \mathrm{u}_k \in \mathrm{Ker}\, D.$$

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Commuting flows

The discussion of Wilson on commuting flows can be taken over almost verbatim.

Definition : An evolutionary derivation ∂ is a map on $\mathcal{R}_c$, which commutes with D:

$$\partial D\mathrm{a} = D\partial\mathrm{a}$$

By setting $\partial D = 0$ this map ∂ extends to $\mathcal{R}_c\{D\}$ such that with $ad_D X = [D, X]$ the relation $[\partial, ad_D] = 0$ is valid.

Definition: For given $\mathbf{X} = \sum_{-\infty}^{d} \mathrm{x}_n D^n$

$$\mathcal{R}_{\mathrm{Ker}\,D,\mathbf{X}} = \mathrm{Ker}\,D[\mathrm{x}_d, \mathrm{x}_{d-1}, \cdots, D\,\mathrm{x}_d, D\,\mathrm{x}_{d-1}, \cdots, D^2\,\mathrm{x}_d, D^2\,\mathrm{x}_{d-1}, \cdots]$$

is the polynomial algebra over Ker D generated by the x_k and their derivatives.

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Commuting flows

Theorem: If the coefficient of the leading term of $\mathbf{X}$ equals one, then

$$\mathcal{Z}_{\mathcal{R}_{\mathrm{Ker}\,D,\mathbf{X}}\{D\}}(\mathbf{X}) = \mathcal{Z}_{\mathcal{R}_c\{D\}}(\mathbf{X}).$$

Definition: For given $\mathbf{X} = \sum_{-\infty}^{d} x_n D^n$ the expression $\mathbf{X}_+ = \sum_0^d x_n D^n$ is called its differential operator part.

For any $\mathbf{P} \in \mathcal{Z}_{\mathcal{R}_c\{D\}}(\mathbf{X})$, define the evolutionary derivation $\partial_{\mathbf{P}}$ on the ring $\mathcal{R}_{\mathrm{Ker}\,D,\mathbf{X}}$ by $\partial_{\mathbf{P}} x_k =$ coefficient of D^k in $[\mathbf{P}_+, \mathbf{X}]$.

The following is an almost verbatim transcription of a result by Wilson

Theorem: Let a Lax operator $\mathbf{L}$ be given. For any $\mathbf{P}, \mathbf{Q} \in \mathcal{Z}_{\mathcal{R}_c\{D\}}(\mathbf{L})$ the derivations $\partial_{\mathbf{P}}$ and $\partial_{\mathbf{Q}}$ on $\mathcal{R}_{\mathrm{Ker}\,D,\mathbf{L}}\{D\}$ commute, $[\partial_{\mathbf{P}}, \partial_{\mathbf{Q}}] = 0$. Any $\partial_{\mathbf{P}}$ commutes with the derivation ad_D.

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The

Lax pair

for the **KdV flow** is given by the Lax operator itself and a special choice of the evolutionary derivation

$$\mathbf{L} = \mathbf{L}_{\mathcal{M},\mathrm{q}} = D_{\mathcal{M}}^2 + \mathrm{q}$$

$$\mathbf{B} = \mathbf{B}_{\mathcal{M},\mathrm{q}} = 4[\mathbf{L}]_+^{3/2} = 4D_{\mathcal{M}}^3 + 3\left(D_{\mathcal{M}}\,\mathrm{q} + \mathrm{q}\,D_{\mathcal{M}}\right)$$

$D_{\mathcal{M}}$ denotes the self adjoint Dirac operator given by the boundary condition $\mathcal{M} = \mathcal{M}(A, B)$ discussed above.

KdV flow

Theorem: If $\mathrm{q} \in \mathcal{R}_c$ is hermitian, then both $\mathbf{L}_{\mathcal{M},\mathrm{q}}$ and $\mathbf{B}_{\mathcal{M},\mathrm{q}}$ are selfadjoint on the Hilbert space of square integrable forms. If $\mathbf{L}_{\mathcal{M}(t),\mathrm{q}(t)}$ is a solution to the Lax equation

$$\mathrm{i}\partial_t \mathbf{L}_{\mathcal{M}(t),\mathrm{q}(t)} = \left[\mathbf{B}_{\mathcal{M}(t),\mathrm{q}(t)}, \mathbf{L}_{\mathcal{M}(t),\mathrm{q}(t)}\right]$$

then the isospectral property

$$\mathbf{L}_{\mathcal{M}(t),\mathrm{q}(t)} = V_{\mathcal{M}(\cdot),\mathrm{q}(\cdot)}(t)\mathbf{L}_{\mathcal{M}(t=0),\mathrm{q}(t=0)} V_{\mathcal{M}(\cdot),\mathrm{q}(\cdot)}(t)^{-1}$$

holds where the unitary $V_{\mathcal{M}(\cdot),\mathrm{q}(\cdot)}(t)$ satisfies

$$\mathrm{i}\partial_t V_{\mathcal{M}(\cdot),\mathrm{q}(\cdot)}(t) = \mathbf{B}_{\mathcal{M}(t),\mathrm{q}(t)} V_{\mathcal{M}(\cdot),\mathrm{q}(\cdot)}(t)$$

with initial condition $V_{\mathcal{M}(\cdot),\mathrm{q}(\cdot)}(t=0) = \mathbb{I}$.

Things to be done (Desiderata)

1. Improved understanding of the role of boundary conditions. In particular in the KdV case: In analogy to the flow of the potential $q(t)$, is there a flow $\mathcal{M}(t)$ in the space of boundary conditions?
2. A construction of a symplectic structure [1]
3. A discussion of the conserved quantities
4. Can one use inverse methods from quantum scattering theory to solve the evolution equations?
5. Discuss further examples: Nonlinear Schrödinger equation (NLS)
6. Find explicit solitary wave solutions
7. Find explicit kink solutions

[1] The gate from the Gel'fand-Dikii formulation to the sympectic formulation of the standard KdV equations was given by M. Adler. In the present context we have so far not been able to find the corresponding key.

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