Integral formulations of the geometric eikonal equation

Aurélien Monteillet

Torino, July 4th 2006

1 – Front propagation

1.1 – Level-set method.

We are interested in front propagations governed by the law

$$V_{t,x} = c(t, x) \tag{0.1}$$

where $V_{t,x}$ denotes the normal velocity of the point x of the front at time t.

Références

[1] Alvarez, O. ; Cardaliaguet, P. ; Monneau, R., *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, *to appear in* Interfaces and free boundaries.

[2] Barles, G. ; Ley, O., *Nonlocal first-order Hamilton-Jacobi equations modelling dislocations dynamics*, *to appear in* Comm. Partial Differential Equations.

[3] Barles, G. ; Soner, H.M. ; Souganidis, P.E., *Front propagation and phase field theory*, SIAM J. Control Optim. 31 (1993), no. 2, 439-469.

1 – Front propagation

Assume that the front $\Gamma(t) = \partial\Omega(t)$ is smooth, and that there exists a smooth $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\Omega(t) = \{x \in \mathbb{R}^N ;\, u(t, x) > 0\}, \quad \Gamma(t) = \{x \in \mathbb{R}^N ;\, u(t, x) = 0\}$$

and $Du(t, x) \neq 0$ when $x \in \Gamma(t)$, where Du is the gradient of u with respect to x.

Then $V_{t,x} = \frac{u_t(t,x)}{|Du(t,x)|}$ and u therefore satisfies the *eikonal* equation :

$$u_t(t, x) = c(t, x)|Du(t, x)|. \tag{0.2}$$

1 – Front propagation

To generalize the preceding evolution to non-smooth fronts, we realize the following program :

1. Find $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $\Gamma(0) = \{x \in \mathbb{R}^N ;\, u_0(x) = 0\}$

2. Solve in an appropriate sense the problem

$$\begin{cases} u_t(t,x) = c(t,x)|Du(t,x)| & \text{for } (t,x) \in (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^N \end{cases} \tag{0.3}$$

3. Set $\Gamma(t) = \{x \in \mathbb{R}^N ;\, u(t,x) = 0\}$

1 – Front propagation

Theorem 0.1 (M. Crandall, P.L. Lions)**.**

Under the following assumptions :

(H) $c : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *is continuous, bounded, Lipschitz continuous with respect to the second variable,*

the problem

$$\begin{cases} u_t(t,x) = c(t,x)|Du(t,x)| & \textit{for}\,(t,x) \in (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \textit{for}\,x \in \mathbb{R}^N \end{cases} \tag{0.4}$$

has a unique uniformly continuous viscosity solution on $[0,T] \times \mathbb{R}^N$ *for all initial datum* u_0 *that is uniformly continuous on* $\mathbb{R}^N$*.*

1 – Front propagation

From now on, we restrict ourselves to sub-solutions.

A result from Barles, Soner and Souganidis ([3]) shows that any sub-solution u of a geometric equation gives another sub-solution $1_{\{u\geq 0\}}$.

We infer, for any closed set $K(0)\subset\mathbb{R}^N$, that there exists a family $(K(t))_{t\in[0,T]}$ such that

1. The graph of K, $\displaystyle\bigcup_{t\in[0,T]}\{t\}\times K(t)$ is closed in $\mathbb{R}^{N+1}$.

2. $(t,x)\mapsto 1_{K(t)}(x)$ is a sub-solution of the eikonal equation on $(0,T)\times\mathbb{R}^N$.

1 – Front propagation

Proposition 0.2. *If $K(0)$ is compact, the evolution is bounded : there exists $R > 0$ such that for all $t \in [0, T]$, $K(t) \subset B(0, R)$.*

2 – Hadamard's formula

If $\Gamma(t) = \partial\Omega(t)$ is a smooth hypersurface of $\mathbb{R}^N$ for $t \geq 0$, and $(\Gamma_t)_{t\geq 0}$ evolves smoothly in time, *Hadamard's formula* states that :

For all $\phi \in C^1([0,+\infty) \times \mathbb{R}^N)$,

$$\frac{d}{dt}\int_{\Omega(t)} \phi(t,x)\,dx = \int_{\Omega(t)} \frac{\partial\phi}{\partial t}(t,x)\,dx + \int_{\partial\Omega(t)} V_{t,x}\,\phi(t,x)\,d\mathcal{H}^{N-1}(x) \tag{0.5}$$

2 – Hadamard's formula

In particular if $V_{t,x} \leq c(t,x)$ in the classical sense, we have for all $\phi \in C^1([0,+\infty) \times \mathbb{R}^N, \mathbb{R}_+)$:

$$\frac{d}{dt} \int_{\Omega(t)} \phi(t,x)\, dx \leq \int_{\Omega(t)} \frac{\partial \phi}{\partial t}(t,x)\, dx + \int_{\partial\Omega(t)} c(t,x)\, \phi(t,x)\, d\mathcal{H}^{N-1}(x) \tag{0.6}$$

2 – Hadamard's formula

When $(t,x) \mapsto 1_{K(t)}(x)$ is "only" a viscosity sub-solution of the eikonal equation, we don't know anything about the regularity of $K(t)$, and the term $\int_{\partial K(t)} c(t,x)\,\phi(t,x)\,d\mathcal{H}^{N-1}(x)$ does not make any sense.

2 – Hadamard's formula

When $(t,x) \mapsto 1_{K(t)}(x)$ is "only" a viscosity sub-solution of the eikonal equation, we don't know anything about the regularity of $K(t)$, and the term $\int_{\partial K(t)} c(t,x)\,\phi(t,x)\,d\mathcal{H}^{N-1}(x)$ does not make any sense.

However, under the conditions

i. $c(t,x) > 0 \ \forall (t,x) \in [0,T] \times \mathbb{R}^N$,

ii. $K(0)$ has the interior ball property of radius $r > 0$, *i.e.* is the union of closed balls of radius $r > 0$,

it has been proved by O. Alvarez, P. Cardaliaguet and R. Monneau that $K(t)$ also satisfies an interior ball condition, and that Hadamard's formula still holds.

2 – Hadamard's formula

An essential point is the following theorem that gives a control on the perimeter of sets with the interior ball property :

Theorem 0.3. *For all $r > 0$, there exists $M > 0$ such that for all closed set $E \subset \mathbb{R}^N$ having the interior ball property of radius r and of diameter less than $1/r$, we have*

$$\mathcal{H}^{N-1}(\partial E) \leq M.$$

2 – Hadamard's formula

From here on, we do not make any assumption either on the sign of c, nor on the regularity of the initial set.

We only consider a family $(K(t))_{t\in[0,T]}$ with closed graph such that $(t,x)\mapsto 1_{K(t)}(x)$ is a sub-solution of the eikonal equation on $(0,T)\times\mathbb{R}^N$.

Set for all $\varepsilon>0$,

$$K^\varepsilon(t)=\{x\in\mathbb{R}^N;\, d_{K(t)}(x)<\varepsilon\}$$

Then $\overline{K^\varepsilon(t)}$ has the interior ball property.

$\rightarrow$ We want to generalize Hadamard's formula to the evolution $t\rightarrow K^\varepsilon(t)$.

2 – Hadamard's formula

To this end, we will have to modify the equation, which leads to introduce a perturbed velocity

$$c^{\varepsilon}(t,x) = \max_{|y-x|\leq\varepsilon} c(t,y)$$

3 – The integral formulation

3.1 – Statement of the result

Theorem 0.4. *Let* $K : [0,T] \to \mathcal{P}(\mathbb{R}^N) \setminus \{\emptyset\}$ *be such that*

1. $K(0)$ *is compact and* $K(t) \to K(0)$ *in the Hausdorff distance as* $t \to 0$,

2. $\bigcup_{t\in[0,T]} \{t\} \times K(t)$ *is closed in* $\mathbb{R}^{N+1}$,

3. $u : (t,x) \mapsto 1_{K(t)}(x)$ *is a viscosity sub-solution of the eikonal equation.*

3 – The integral formulation

Then for all t_1 et t_2 satisfying $0 \leq t_1 < t_2 \leq T$, for almost all $\varepsilon > 0$, and for all $\phi \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R}_+)$,

$$\int_{t_1}^{t_2} \int_{K^\varepsilon(s)} \phi_t(s,x)\, dxds + \int_{t_1}^{t_2} \int_{\partial K^\varepsilon(s)} c^\varepsilon(s,x)\, \phi(s,x)\, d\mathcal{H}^{N-1}(x) ds$$

$$\geq \left[\int_{K^\varepsilon(s)} \phi(s,x)\, dxds \right]_{t_1}^{t_2} \tag{0.7}$$

3 – The integral formulation

3.2 – Steps of proof

Let $w(t,x) = -d_{K(t)}(x)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be smooth and non-decreasing such that $\theta = 0$ in $(-\infty, -\varepsilon]$, $\theta = 1$ in $[0, \infty)$, and set $w_\theta = \theta \circ w$.

3 – The integral formulation

3.2 – Steps of proof

Let $w(t,x) = -d_{K(t)}(x)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be smooth and non-decreasing such that $\theta = 0$ in $(-\infty, -\varepsilon]$, $\theta = 1$ in $[0, \infty)$, and set $w_\theta = \theta \circ w$.

1. w_θ is a sub-solution of $(w_\theta)_t = c^\varepsilon(t,x)|Dw_\theta|$ in $(0,T) \times \mathbb{R}^N$.

3.2 – Steps of proof

Let $w(t,x) = -d_{K(t)}(x)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be smooth and non-decreasing such that $\theta = 0$ in $(-\infty, -\varepsilon]$, $\theta = 1$ in $[0, \infty)$, and set $w_\theta = \theta \circ w$.

1. w_θ is a sub-solution of $(w_\theta)_t = c^\varepsilon(t,x)|Dw_\theta|$ in $(0,T) \times \mathbb{R}^N$.

2. For all $\phi \in C^1_c((0,T) \times \mathbb{R}^N, \mathbb{R}_+)$,

$$\int_0^T \int_{\mathbb{R}^N} w_\theta(s,x)\,\phi_t(s,x)\,dxds + \int_0^T \int_{\mathbb{R}^N} c^\varepsilon(s,x)|Dw_\theta(s,x)|\,\phi(s,x)\,dxds \geq 0$$

(this is obtained by a regularization in time and integration by parts).

3 – The integral formulation

As θ tends to $1_{(-\varepsilon,+\infty)}$:

3. $\int_0^T \int_{\mathbb{R}^N} w_\theta(s,x)\,\phi_t(s,x)\,dxds \to \int_0^T \int_{K^\varepsilon(s)} \phi_t(s,x)\,dxds.$

3 – The integral formulation

As θ tends to $1_{(-\varepsilon,+\infty)}$:

3. $\int_0^T \int_{\mathbb{R}^N} w_\theta(s,x)\, \phi_t(s,x)\, dxds \to \int_0^T \int_{K^\varepsilon(s)} \phi_t(s,x)\, dxds.$

4. The coarea formula shows that

$$\begin{aligned} &\int_{\mathbb{R}^N} c^\varepsilon(s,x)|Dw_\theta(s,x)|\, \phi(s,x)\, dx \\ &= \int_0^1 \int_{\{w_\theta(s,\cdot)=\tau\}} c^\varepsilon(s,x)\, \phi(s,x)\, d\mathcal{H}^{N-1}(x) d\tau \\ &\to \int_{\partial K^\varepsilon(s)} c^\varepsilon(s,x)\, \phi(s,x)\, d\mathcal{H}^{N-1}(x) \quad \text{for a.a. } \varepsilon > 0. \end{aligned}$$

3 – The integral formulation

Conclusion :

For almost all $\varepsilon > 0$, for all $\phi \in C_c^1((0,T) \times \mathbb{R}^N, \mathbb{R}_+)$,

$$\int_0^T \int_{K^\varepsilon(s)} \phi_t(s,x)\, dxds + \int_0^T \int_{\partial K^\varepsilon(s)} c^\varepsilon(s,x)\, \phi(s,x)\, d\mathcal{H}^{N-1}(x) ds \geq 0.$$

4 – The converse theorem

The following theorem shows that the integral formulation characterizes the fact for $(t, x) \mapsto 1_{K(t)}(x)$ to be a viscosity sub-solution of the eikonal equation :

4 – The converse theorem

The following theorem shows that the integral formulation characterizes the fact for $(t,x) \mapsto 1_{K(t)}(x)$ to be a viscosity sub-solution of the eikonal equation :

Theorem 0.5. *Let* $K : [0,T] \to \mathcal{P}(\mathbb{R}^N) \setminus \{\emptyset\}$ *be such that*

1. $\bigcup_{t\in[0,T]}\{t\} \times K(t)$ *is closed in* $\mathbb{R}^{N+1}$ *and* K *is bounded.*

2. For almost all small enough $\varepsilon > 0$, *for all* $\phi \in C^1_c((0,T) \times \mathbb{R}^N, \mathbb{R}_+)$,

$$\int_0^T \int_{K^\varepsilon(s)} \phi_t(s,x)\, dxds + \int_0^T \int_{\partial K^\varepsilon(s)} c^\varepsilon(s,x)\, \phi(s,x)\, d\mathcal{H}^{N-1}(x)ds \geq 0.$$

Then $u : (t,x) \mapsto 1_{K(t)}(x)$ *is a viscosity sub-solution of* $u_t = c(t,x)|Du|$ *in* $(0,T) \times \mathbb{R}^N$.

5 – Regularity of the front

5.1 – BV functions and sets of finite perimeter

Let Ω be an open subset of $\mathbb{R}^N$.

Definition 0.6. An application $f \in L^1_{loc}(\Omega)$ is said to have locally bounded variations in Ω if for all open set $U \subset\subset \Omega$,

$$\sup\{\int_U f(x)\, div\, \phi(x)\, dx\, ;\, \phi \in C^1_c(U, \mathbb{R}^N)\, ;\, \|\phi\|_\infty \leq 1\} < +\infty.$$

We denote by $BV_{loc}(\Omega)$ the set of functions of locally bounded variations in Ω.

Likewise, we say that $f \in L^1(\Omega)$ has bounded variations in Ω (BV) if the preceding definition holds for $U = \Omega$. We denote by $BV(\Omega)$ their set.

5 – Regularity of the front

From the Riesz representation theorem, we deduce :

Theorem 0.7. *Let* $f \in BV_{loc}(\Omega)$. *Then there exists a Radon measure* μ *on* Ω *and a* μ*-measurable application* $\sigma : \Omega \to \mathbb{R}^N$ *such that :*

1. $|\sigma(x)| = 1$ μ*-a.e.*

2. $\int_\Omega f(x)\, div\, \phi(x)\, dx = -\int_\Omega \langle \phi(x), \sigma(x) \rangle\, d\mu \quad \forall \phi \in C_c^1(\Omega, \mathbb{R}^N)$.

The measure μ *is called the variation measure of* f, *and is denoted by* $\|Df\|$.

Definition 0.8. A $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$ is said to have (locally) finite perimeter in Ω if 1_E has (locally) bounded variations in Ω.

The variation measure of 1_E is is this case denoted $\|\partial E\|$, and the function $-\sigma$ given by theorem 0.7 is denoted ν_E.

We thus have for all $\phi \in C_c^1(\Omega, \mathbb{R}^N)$,

$$\int_E div\, \phi(x)\, dx = \int_{\mathbb{R}^N} \langle \phi(x), \nu_E(x) \rangle\, d\|\partial E\|. \tag{0.8}$$

5 – Regularity of the front

Definition 0.9. Let E be a set of locally finite perimeter in Ω. We say that $x \in \Omega$ belongs to the reduced boundary of E, denoted $\partial^* E$, if :

1. $\|\partial E\|(B(x,r)) > 0 \quad \forall r > 0$,
2. $\frac{1}{\|\partial E\|(B(x,r))} \int_{B(x,r)} \nu_E(y)\, d\|\partial E\| \underset{r \to 0}{\longrightarrow} \nu_E(x)$,
3. $|\nu_E(x)| = 1$.

Then we have the following result :

Theorem 0.10. *Let E be a set of locally finite perimeter in Ω.*
Then :

1. $\|\partial E\|(B) = \mathcal{H}^{N-1}(B \cap \partial^* E)$ *for all Borel set* $B \subset \Omega$.

2. Gauss-Green formula : For all $\phi \in C_c^1(\Omega, \mathbb{R}^N)$,

$$\int_E div\, \phi(x)\, dx = \int_{\partial^* E} \langle \phi(x), \nu_E(x) \rangle \, d\mathcal{H}^{N-1}(x). \tag{0.9}$$

5 – Regularity of the front

5.2 – Perimeter estimate

Theorem 0.11. *Let* $K : [0, T] \to \mathcal{P}(\mathbb{R}^N)$ *with closed graph in* $\mathbb{R}^{N+1}$ *be such that*

1. $K(0)$ *is compact,*

2. $K(t) \to K(0)$ *in the Hausdorff distance as* $t \to 0$,

3. $u : (t, x) \mapsto 1_{K(t)}(x)$ *is a viscosity sub-solution of the eikonal equation. Under the additional assumptions*

(A1) c *is of class* C^1, Dc *is locally Lipschitz continuous with respect to the second variable,*

(A2) $Dc(t, x) \neq 0$ *if* $c(t, x) = 0$,

5 – Regularity of the front

We have :

1. For almost all $t \in [0,T]$, $c(t,\cdot)\, 1_{K(t)}$ has bounded variations in $\{c(t,\cdot) < 0\}$.

2. For almost all $t \in [0,T]$, $K(t)$ has locally finite perimeter in $\{c(t,\cdot) < 0\}$.

3. If we denote $(\cdot)_-$ the negative part of a quantity $((x)_- = \max(-x,0))$, we have :

$$\int_0^T \int_{\partial^* K(s)} c_-(s,x)\, d\mathcal{H}^{N-1}(x) ds < +\infty.$$

5 – Regularity of the front

Heuristic idea : apply the integral formulation with $\varepsilon = 0$ to $\phi = 1_{\{c<0\}}$:

$$\int_0^T \int_{K(s)} \phi_t(s,x)\,dxds + \int_0^T \int_{\partial K(s)} c(s,x)\,\phi(s,x)\,d\mathcal{H}^{N-1}(x)ds$$

$$\geq \left[\int_{K(s)} \phi(s,x)\,dxds \right]_0^T$$

5 – Regularity of the front

1. $\left[\int_{K(s)} \phi(s,x)\, dxds\right]_0^T = \left[\int_{K(s)} 1_{\{c<0\}}(s,x)\, dxds\right]_0^T.$

5 – Regularity of the front

1. $\left[\int_{K(s)} \phi(s,x)\,dxds\right]_0^T = \left[\int_{K(s)} 1_{\{c<0\}}(s,x)\,dxds\right]_0^T.$

2. $\int_0^T \int_{K(s)} \phi_t(s,x)\,dxds \leq \int_0^T \int_{K(s)\cap\{c(s,\cdot)=0\}} \frac{|c_t(s,x)|}{|Dc(s,x)|}\,d\mathcal{H}^{N-1}(x)ds.$

5 – Regularity of the front

Thank you!